Formulas for Shaft Friction. By J. J. Flather.

Among the various methods employed for the long distance transmission of power shafting has been used to a limited extent.

In many of the earlier applications the motion was one of translation. Thus in the transmission of power from the large overshot wheel at Laxey, on the Isle of Man, trussed rods are used to transmit about 150 h. p. several hundred feet; the rods are continuously connected and are supported on wheel carriers running on iron ways.

This method was adopted, on a very large scale, in the mines of Devonshire for the transmission of power from large overshot water wheels to pumps fixed in the shaft of the mine at a considerable distance higher up the valley.

In one case the water wheel was 52 feet diameter, 12 feet breast, and its ordinary working speed was 5 revolutions per minute. The length of stroke given by the crank to the horizontal or "flat" rods was 8 feet; the rods were 3½-inch round iron, and were carried on cast-iron pulleys.

At Devon Great Consols, near Tavistock, there are altogether very nearly three miles of 3-inch wrought-iron rods, carried on bobs, pulleys and stands, whereby power for pumping and winding is conveyed along the surface to different parts of these extensive mines from 11 large water wheels ranging up to 50 feet in diameter.

In the transmission of power by rotating shafting supported in bearings throughout its length, the friction of the journals is a very important consideration, and effectually debars its use for long-distance transmission.

This can be seen in the following formulæ, which show the relation between the horse-power required to overcome the friction of the shaft due to its weight and velocity, and the horse-power transmitted by the shaft for a given diameter and length corresponding to an angular distortion of $\frac{1}{10}$ degree per foot of length.

If the contact between shaft and its bearing be a line contact only, the initial load which produces friction will be P; on the other hand, if the shaft exactly fits the bearing the friction load will be $\frac{\pi}{2}$ P; midway between these lies a value, P×1.28, or $\frac{4}{\pi}$ P, which will be here assumed as closely approaching conditions of actual practice when the journal is well worn to its bearing.

Under these conditions the friction horse-power will be:

$$H.P_{0}. = \frac{F\,v}{33000} = \frac{4}{\pi}\,o\,\frac{W \times v}{33000} \qquad (1)$$

In which F = load due to friction;
v = velocity of surface of shaft;
φ = coefficient of friction for factory shafting;
W = weight of shaft.

While φ varies from 0.03 to 0.08 under different conditions, we have assumed it to equal 0.06 for ordinary factory shafting, with more or less imperfect lubrication and alignment.

If there are no pulleys on the shaft, W will equal

$$\frac{\pi}{4} d^2 L \times 3.36 \text{ pounds, where}$$

L = length of shaft in feet, and
d = diameter of shaft in inches.

The horse-power exerted to overcome friction will then be:

$$H. P_0. = \frac{F v}{33000} = \frac{4}{\pi} \varphi \times \frac{\pi}{4} \frac{d^2 L \times 3.36 v}{33000} = 0.000006 d^2 L v. \qquad (2)$$

The horse-power transmitted by the shaft will be:

$$H. P. = \frac{\pi}{16} \frac{d^3 f \times 2 \pi N}{12 \times 33000}. \qquad (3)$$

If we assume the angle of torsion not to exceed $\frac{1}{10}$ degree per foot length of shaft, there is obtained

$$\theta^0 = \frac{360}{2\pi} \times f \frac{L \times 12}{G r} = \frac{360 f}{\pi G} \times \frac{12 L}{d}; \qquad (4)$$

hence:

$$f = \frac{0.10 L \times \pi G d}{360 \times 12 L} = 800 d$$

when G — 11,000,000; that is, when the modulus of torsion = $\frac{2}{5}$ modulus of elasticity.

Substituting this value of f in (3), and noting that $v = \frac{\pi d N}{12}$, we have:

$$H. P. = 0.0095 d^3 v. \qquad (5)$$

From equations (2) and (5) there is obtained

$$\frac{H. P_0.}{H. P.} = \frac{0.000006 d^2 L v}{0.0095 d^3 v};$$

$$\text{that is, } H. P_0. = 0.000063 H. P. \frac{L}{d} = \frac{L}{d} \times \frac{H. P.}{1600} \text{ very closely.} \qquad (6)$$

We see from this that the horse-power required to overcome the friction of a one-inch shaft 1600 feet long is equal to the total allowable transmitting capacity of the shaft under ordinary working conditions.

The following table, calculated from this formula, gives the limits in which the power transmitted by a shaft would be absorbed by the friction of the bearings under the above conditions:

Diameter of Shaft in Inches.	Length in Feet When Total Power is Absorbed.	Length When $\eta = 50$ per cent.	Length When $\eta = 75$ per cent.
1	1600	800	400
2	3200	1600	800
3	4800	2400	1200
4	6400	3200	1600
5	8000	4000	2000

In the ordinary transmission of power by shafting we find the shaft loaded with pulleys and the power taken off by varying amounts throughout its entire length. It is unusual, except in short lengths, to receive the power at one end and transmit it at the other. Moreover, in long shafting the head, or receiving shaft, is usually situated midway between the ends, and the power distributed more or less uniformly from this headshaft to either end; therefore, in estimating the power absorbed by friction in ordinary mill or factory shafting loaded with pulleys, the previous formulæ do not apply, as these relate only to those cases where power is taken off at the end of the shaft.

The conditions of practice, as we find them in actual transmissions are so various, that it is difficult to lay down any general rule by which the power absorbed by friction may be determined. The number and weight of pulleys and couplings, the intensity and direction of belt-pull, the condition of bearings and their lubrication; these all affect the amount of work lost in friction.

For the ordinary factory shafting, from which power is taken fairly uniformly throughout its length and distributed horizontally to counter or auxiliary shafts situated on one or both sides of the main shaft, there will be three general cases to be considered, and each of these will be modified, depending upon the direction of the belt to and from the main shaft.

The friction will evidently be proportional to the weight of the shaft and the unbalanced belt-pull acting on the shaft.

The weight of pulleys, belts, clutches and couplings carried by the line shaft will vary from about one and one-half to three times the weight of shaft, so that the total weight on the bearings will vary from two and one-half to four times the weight of shaft; for head and jack shafts the total weight will probably vary from three to five times the weight of shaft.

In addition to this weight there is the unbalanced belt-pull which increases the load on the bearings. Although the tension on the tight side of the belt may not ordinarily exceed about twice the tension in the slack side necessary for adhesion, yet it is probable that belts are frequently run with a ratio of tension equal to one to three, and occasionally one to four. On the other hand, it is a very common thing for belts, especially short ones, to be laced so taut that the initial tension is greatly in excess of that required for adhesion, in which case the sum of the tensions approaches twice that in the tight side of the belt.

With ordinary shop-worn belting it will be safe to assume that the tension T_2 on the slack side of the belts is one-half the tension T_1 on the tight or driving side, that is $T_2 = \frac{T_1}{2}$, hence, since $T_1 - T_2 = P$, the driving force, we have

$$H.\ P. = \frac{T_1}{2} \times \frac{V}{33000}. \qquad (7)$$

Under the conditions which obtain in machine shops the diameter of a shaft to safely transmit a given horse-power without undue deflection may be obtained from the formula

$$d = \sqrt[3]{\frac{H.\ P.}{N}} \times 100. \qquad (8)$$

Combining (7) and (8) we have

$$H.\ P. = \frac{T_1}{2} \times \frac{V}{33000} = \frac{d^3\ N}{100}, \qquad (9)$$

$$\text{and } T_1 = \frac{660\ d^3\ N}{V}.$$

Therefore the sum of the tensions on the entire length of shaft

$$= \Sigma\,(T_1 + T_2) = \frac{3}{2}\, d^3\ N \times \frac{660}{V}, \qquad (10)$$

$$\text{or } B_1 = \frac{1000}{V}\, d^3\ N \text{ very nearly.}$$

Hence the belt-pull per foot of length of shaft $= 1000 \frac{d^3}{L} \frac{N}{V}$. The force of friction, F, acting at the circumference of shaft, is $\frac{4}{\pi} \varphi W$, as before, but in this case W equals the weight of shaft, W_0 and its furniture, as well as the unbalanced belt-pull.

The belt tension may act in any direction perpendicular to the axis of the shaft, and the intensity of pull in any given direction will vary from 0 to the maximum to total tension. Besides these tensions there will be an additional pull due to the tensions in the belt from fly-wheel to main line shaft.

Let B_1=belt-pull due to total tensions acting at an angle, β, with the horizontal;

B^1=belt-pull due to tensions in main belt acting at an angle, α, with the horizontal;

V^1=linear velocity of main belt from fly-wheel;

V=average linear velocity of cross belts;

$r=\frac{V^1}{V}$=ratio of velocity of main belt to average velocity of cross belts,

then the horizontal pull$=B^1 \cos\alpha + B_1 \cos\beta$ (11)

and the vertical pull$=B^1 \sin\alpha + B_1 \sin\beta$ (12)

But $B^1 = \frac{B_1}{r} = 1000 \frac{d^3N}{Vr}$; (13)

therefore, the horizontal pull$=B_1\left(\frac{\cos\alpha}{r} + \cos\beta\right)=x$;

and the vertical pull$=B^1\left(\frac{\sin\alpha}{r} + \sin\beta\right)=y$;

If $\alpha=O$ and $\beta=O$,

then $x=B_1\left(\frac{1}{r} + 1\right)$

and $y=O$.

The most usual case, when the power is not taken off equally on either side, will be that in which main belt makes an angle with the horizontal, and the cross belts are themselves horizontal, that is:

$$x=B_1\left(\frac{\cos\alpha}{r} + 1\right)$$

$$y=\frac{B_1}{r}\sin\alpha$$

When the horizontal cross-belts are distributed equally on either side of the shaft the only load we need consider will be that due to the main belt, in which case

$$x=\frac{B_1\cos\alpha}{r}, \text{ and}$$

$$y=\frac{B_1\sin\alpha}{r}.$$

If the machines be driven from below, the pull of the belts, instead of adding to the load on the bearings, will cause this load to be decreased; but as this method is not usual we shall not consider it here.

Combining the load on the shaft due to the belt pull with that due to its weight, the resultant load will be

$$\sqrt{x^2 + (W_s + y)^2}, \qquad (14)$$

hence the friction load will be

$$F = \frac{4}{\pi} \varphi \sqrt{x^2 + (y + W_s)^2},$$

$$\text{If } W_s = 3 \left(\frac{\pi}{4} d^2 \times 3.36 \text{ L}\right) \text{ we have}$$

$$F = \frac{4}{\pi} \varphi \sqrt{x^2 + [y + 3 (\frac{\pi}{4} d^2 \times 3.36 \text{ L})]^2} \qquad (15)$$

Taking a specific case in which the cross belts are assumed to drive horizontally on each side of the line shaft, and the main belt to make an angle of 30° with the horizontal, we have

$$F = \frac{4}{\pi} \varphi \sqrt{\left(\frac{B_1 \cos \alpha}{r}\right)^2 + \left[\frac{B_1 \sin \alpha}{r} + (7.9 d^2 \text{ L})\right]^2}$$

The velocity of intermediate belting is so variable that any assumption of speed must be regarded as applying to a particular case or representative of a certain type of factory, and can not be taken as general. In many machine shops the average speed of intermediate belts is not more than 500 feet per minute; in others the average speed is more than twice as great, and in wood-working shops it is still greater.

For our present purpose we shall assume an average speed of 660 feet per minute for belts running from the main shaft to a secondary or countershaft, and four times this speed for the velocity of belt from engine to main shaft, that is $\frac{V'}{V} = r = 4.$

Substituting these values in (13) we have

$$B_1 = \frac{1}{4} \times \frac{1000 d^3 N}{660} = \frac{1}{4} (\tfrac{3}{2} d^3 N)$$

$$\text{therefore } F = \frac{4}{\pi} \varphi \sqrt{[\tfrac{1}{4} (\tfrac{3}{2} d^3 N \cos \alpha]^2 + [\tfrac{1}{4} (\tfrac{3}{2} d^3 N \sin \alpha) + 7.9 d^2 \text{ L}]^2}$$

$$= \sqrt{[0.025 d^3 N]^2 + [0.014 d^3 N + 0.6 d^2 \text{ L}]^2} \qquad (16)$$

From the formula for the power absorbed by friction we have

$$F v = \text{H. P}_{o}. = \frac{F \pi d N}{33000 \times 12}, \text{ or H. P}_{o}. = 0.0^{5} 8 d NF, \qquad (17)$$

hence the ratio of power absorbed by friction to the horse-power which the shaft is capable of safely transmitting will be

$$\frac{\text{H. P}_{o}.}{\text{H. P.}} = \frac{0.0^{5} 8 d N F}{0.01 d^3 N} = \frac{0.08 F}{d^2} \text{ per cent.}$$

From this expression the following table has been computed for a 3-inch shaft running at 100 and 250 revolutions per minute:

Diameter of shaft in inches.	Revolutions per minute.	Percentage of loss when length in feet.				
		100	200	400	800	1600
3	100	5.1	9.9	19.6	38.7	77
3	250	5.8	10.6	20	39	77½

It is worthy of remark that in long lines of shafting the influence of belt pull on the bearings is very slight compared to the weight of shaft and pulleys, so that the loss in friction is but little more than that due to weight alone.

With better alignment and better lubrication the loss will be less than that here given; in long continuous lines of shafting the bearings are always more or less out of line, and for this reason the loss will be less if short lengths be employed.

Orthogonal Surfaces. By A. S. Hathaway.

It is well known that a given system of surfaces $f(x, y, z) = c$ has in general no pair of orthogonal conjugate systems, *i. e.*, such that the surfaces of the three systems through any point are mutually orthogonal at that point. It has been shown by Cayley [Salmon's Three Dimensions, p. 447] that $f(x, y, z)$ must satisfy a differential equation of third order if it possess a pair of orthogonal conjugates. In the course of some recent investigations on fluid motion I was led to observe that a given system of surfaces might have two pairs of orthogonal conjugates, in which case it would have an infinite number of such pairs. In order that such may be the case $f(x, y, z)$ must satisfy a differential equation of second order which is a particular integral of Cayley's equation of third order. This differential equation is, in Cayley's notation,

$$[(a, b, c, f, g, h)(L, M, N)^2 - (a + b + c)(L^2 + M^2 + N^2)]^2$$
$$= 4(L^2 + M^2 + N^2)(A, B, C, F, G, H)(L, M, N)^2$$

where $L, M, N, a, b, c, f, g, h$, are the first and second differential coefficients of $f(x, y, z)$, and A, B, C, *etc.*, are the minors of a, b, c, *etc.*, in the matrix

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$