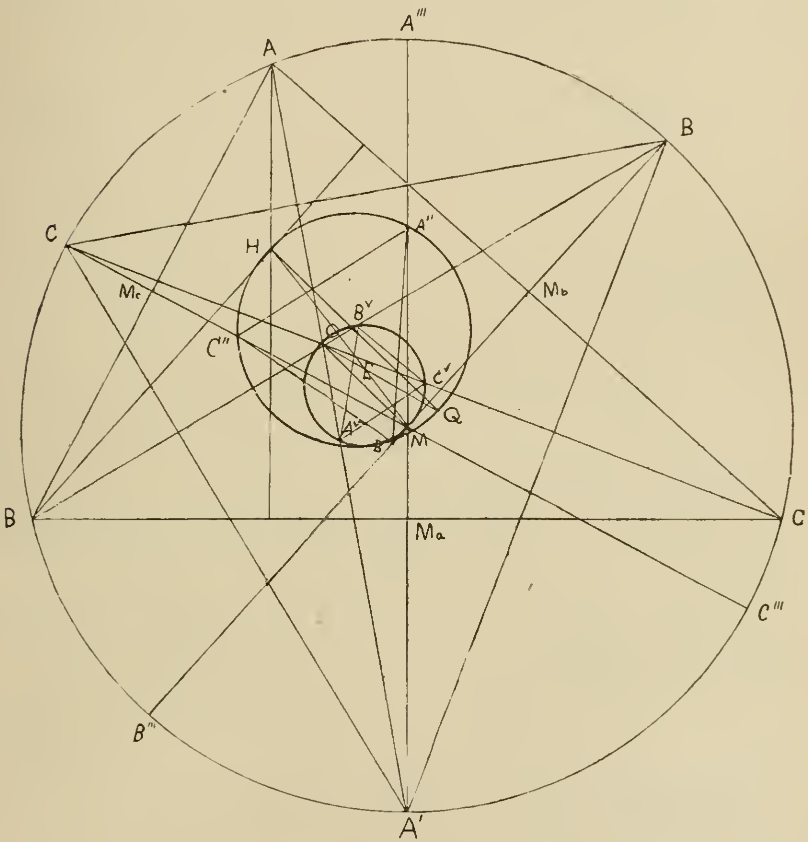
A New Triangle and Some of Its Properties. By Robert Judson Aley.

Explanation of Figure. — ABC is any triangle, of which M is the circumcenter, O the incenter, H the orthocenter, and Q Nagel's Point. $A'A'''$, $B'B'''$ and $C'C'''$ are diameters perpendicular to the sides BC, CA, AB, respectively.



I. If A^{v}, B^{v}, C^{v} are the middle points of AA', BB', CC', respectively, then OM is the diameter of the circumcircle of $A^{\text{v}}B^{\text{v}}C^{\text{v}}$.

Since A^{v} is the middle point of AA',

MA^{v} is parallel to AA'''.

But AA''' is perpendicular to AA',

∴ MA^{v} is perpendicular to AA'.

Hence $MA^{v}O$ is a right angle.

Similarly $MB^{v}O$, and $MC^{v}O$ are right angles.

$\therefore$ a circle upon OM as diameter will pass through A^{v}, B^{v}, C^{v}.

II. The triangle $A^{v}B^{v}C^{v}$ is similar to Nagel's triangle $A''B''C''$.

$\angle B^{v}C^{v}A^{v}$ is the supplement $\angle B^{v}OA^{v}$.

$$\angle B^{v}OA^{v} = \angle AOB.$$
$$= \pi - (\tfrac{1}{2}A + \tfrac{1}{2}B).$$
$$= A + B + C - \tfrac{1}{2}(A + B).$$
$$= C + \tfrac{1}{2}(A + B).$$
$$\pi - \angle B^{v}OA^{v} = \pi - [C + \tfrac{1}{2}(A + B)].$$
$$= A + B + C - [C + \tfrac{1}{2}(A + B)].$$
$$= \tfrac{1}{2}(A + B).$$

$\therefore B^{v}C^{v}A^{v} = \tfrac{1}{2}(A + B)$.

MA^{v} is perpendicular to AA'.

MC^{v} is perpendicular to CC'.

$\angle(MA^{v}, MC^{v}) = \angle(AA', CC')$.

i. e., $\angle A^{v}MC^{v} = A'OC$.

$$= \tfrac{1}{2}(A + C).$$

But $\angle A^{v}MC^{v} = A^{v}B^{v}C^{v}$.

$\therefore \angle A^{v}B^{v}C^{v} = \tfrac{1}{2}(A + C.)$

Similarly $\angle B^{v}A^{v}C^{v} = \tfrac{1}{2}(B + C)$.

The angles of the triangle $A''B''C''$ (Nagel's triangle) are $\tfrac{1}{2}(B + C)$, $\tfrac{1}{2}(A + C)$, $\tfrac{1}{2}(A + B)$, respectively. (Schwatt's Geometric Treatment of Curves, page 39.)

$\therefore A^{v}B^{v}C^{v}$ is similar to $A''B''C''$. It is also similar to $A'B'C'$, for $A'B'C'$ and $A''B''C''$ are similar.

III. O is the centre of perspective of $A^{v}B^{v}C^{v}$ and $A'B'C'$.

IV. E, the centroid of ABC, is the internal center of similitude of the circumscribing circles of $A''B''C''$ and $A^{v}B^{v}C^{v}$.

OM is parallel to HQ.

It is known that H, E, M, are collinear, as are also O, E, Q.

$\therefore HM$ and OQ intersect at E.

$\therefore E$ is the internal center of similitude.

V. E is also the center of perspective of $A^{v}B^{v}C^{v}$ and $A''B''C''$.

For, consider the triangle $AA''A'$.

$A''A^{v}$ is a median and so is AM_a.

$\therefore A''A^{v}$ passes through E.

Now consider the triangle $BB''B'$.

$B''B^v$ is a median and so is BM_b.

$\therefore$ $B''B^v$ passes through E.

In the same way we can show that $C''C^v$ also passes through E.

$\therefore$ E is the center of perspective of $A''B''C''$ and $A^vB^vC^v$.

VI. All the lines in $A^vB^vC^v$ are just one-half the corresponding lines in $A''B''C''$.

This is an immediate consequence of the fact $OM = \frac{1}{2}\ HQ$. (Schwatt, Geomet. Curves, page 40.)

VII. The sides of the triangle $A^vB^vC^v$ are oppositely parallel to the corresponding sides of $A''B''C''$, *i. e.*, A^vB^v is parallel to $B''A''$, etc.

OM is parallel to HQ.

HA'' is perpendicular to AA'.

MA^v is perpendicular to AA'.

$\therefore \angle\ A''HQ = \angle\ OMA^v$.

In the same way

$\angle\ B''HQ = \angle\ OMB^v$.

$\angle\ C''HQ = \angle\ OMC^v$.

This shows that the points A^v, B^v, C^v are located with respect to O, just as A'', B'', C'' are located with respect to Q.

$\angle\ (OM,\ B^vA^v)$ is measured by $\frac{1}{2}$ (arc OB^v + arc A^vC^v + arc C^vM).

$\angle\ (HQ,\ A''B'')$ is measured by $\frac{1}{2}$ (arc $B''Q$ + arc $A''C''$ + arc $C''H$).

But arc OB^v measures the same angle in the circle on OM as diameter, that the arc $B''Q$ measures in the circle on HQ as diameter.

The same is also true of the arcs A^vC^v and $A''C''$, and C^vM and $C''H$.

$\therefore \angle\ (OM,\ B^vA^v) = \angle\ (HQ,\ A''B'')$.

But since OM is parallel to HQ, we have at once $A''B''$ parallel to B^vA^v.

In the same way we may prove that $B''C''$ is parallel to C^vB^v and $C''A''$ parallel to A^vC^v.

$\therefore$ the sides of $A^vB^vC^v$ are oppositely parallel to the corresponding sides of $A''B''C''$.

VIII. The triangle $A^vB^vC^v$ is Nagel's triangle for the triangle $M_aM_bM_c$.

It is known (Schwatt, page 41) that O is Nagel's point in the triangle $M_aM_bM_c$, and that M is the orthocenter. The circle on OM as diameter is Nagel's circle for the triangle $M_aM_bM_c$. We know that the sides of $M_aM_bM_c$ are oppositely parallel to the sides of ABC, and we have proven that $A^vB^vC^v$, inscribed in the Nagel's circle of $M_aM_bM_c$, has its sides oppositely parallel to the sides of Nagel's triangle for ABC.

$\therefore$ $A^vB^vC^v$ is Nagel's triangle for $M_aM_bM_c$.