The Point P and Some of Its Properties.

By Robert J. Aley.

P is the point of concurrence of the lines drawn from the vertices of a triangle to the points of contact of the inscribed circle with the sides. It has been called the Gergonne Point. The ratios of the distances of the point P from the sides are $\frac{1}{a(s-a)} : \frac{1}{b(s-b)} : \frac{1}{c(s-c)}$ (Aley, Contributions to Geom. of the Triangle, p. 10 (10)). From these ratios the actual distances of the point from the sides is easily found to be

$$PP_a = \frac{2\Delta(s-b)(s-c)}{a\Sigma(s-a)(s-b)}$$

$$PP_b = \frac{2\Delta(s-c)(s-a)}{b\Sigma(s-a)(s-b)}$$

$$PP_c = \frac{2\Delta(s-a)(s-b)}{c\Sigma(s-a)(s-b)}$$

P and Q (Nagel's Point) are isotomic conjugates and they are collinear with Z, the isotomic conjugate of I (incentre) (Ibid., page 8, III).

P_1 (the isogonal conjugate of P), Z_1 the isogonal conjugate of Z and K are collinear (Ibid., page 13, IV).

P_1, I and M are collinear.

The ratios of P_1 are $a(s-a) : b(s-b) : c(s-c)$ (Ibid., p. 3, 81).

From these the actual distances of P_1 from the sides is readily found to be

$$P_1P_{1a} = \frac{2\Delta a(s-a)}{S\Sigma a^2 - \Sigma a^3}$$

$$P_1P_{1b} = \frac{2\Delta b(s-b)}{S\Sigma a^2 - \Sigma a^3}$$

$$P_1P_{1c} = \frac{2\Delta c(s-c)}{S\Sigma a^2 - \Sigma a^3} \left\{ \text{Ibid., page 14, (7)} \right\}.$$

It is well known that

$$II_a = \frac{\Delta}{s}$$

$$II_b = \frac{\Delta}{s}$$

$$II_c = \frac{\Delta}{s}$$

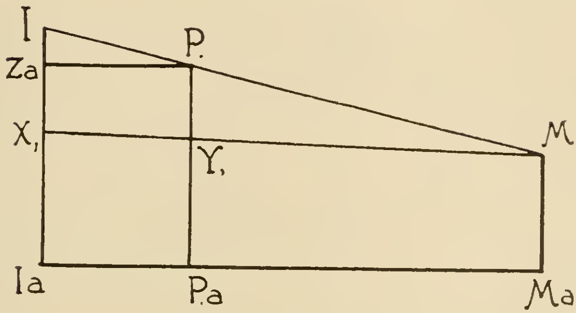
The ratios of M are $a(-a^2+b^2+c^2) : b(a^2-b^2+c^2) : c(a^2+b^2-c^2)$.

From these ratios it is easily found that

$$MM_a = \frac{a(-a^2 + b^2 + c^2)}{8\Delta}$$

$$MM_b = \frac{b\ a^2 - b^2 + c^2}{8\Delta}$$

$$MM_c = \frac{c(a^2 + b^2 - c^2)}{8\Delta}$$



$$IZ = II_a - P_1 P_{1a}$$

$$= \frac{\Delta}{S} - \frac{2\Delta a\ (S-a)}{S\ \Sigma_a^2 - \Sigma_a^3}$$

$$= \frac{\Delta}{S(S\ \Sigma_a^2 - \Sigma_a^3} \left\{ S\ \Sigma_a^2 - \Sigma_a^3 - 2\,a\,S(S-a) \right\}$$

$$= \frac{\Delta}{S(S\ \Sigma_a^2 - \Sigma_a^3)} \left\{ a^2b + a^2c + b^2c + bc^2 - 2\,abc - b^3 - c^3 \right\}$$

$$IX = II_a - MM_a$$

$$= \frac{\Delta}{S} - \frac{a(-a^2 - b^2 + c^2)}{8\Delta}$$

$$= \frac{1}{8S\Delta} \left\{ 8\Delta^2 - a\,S(-a^2 + b^2 + c^2) \right\}$$

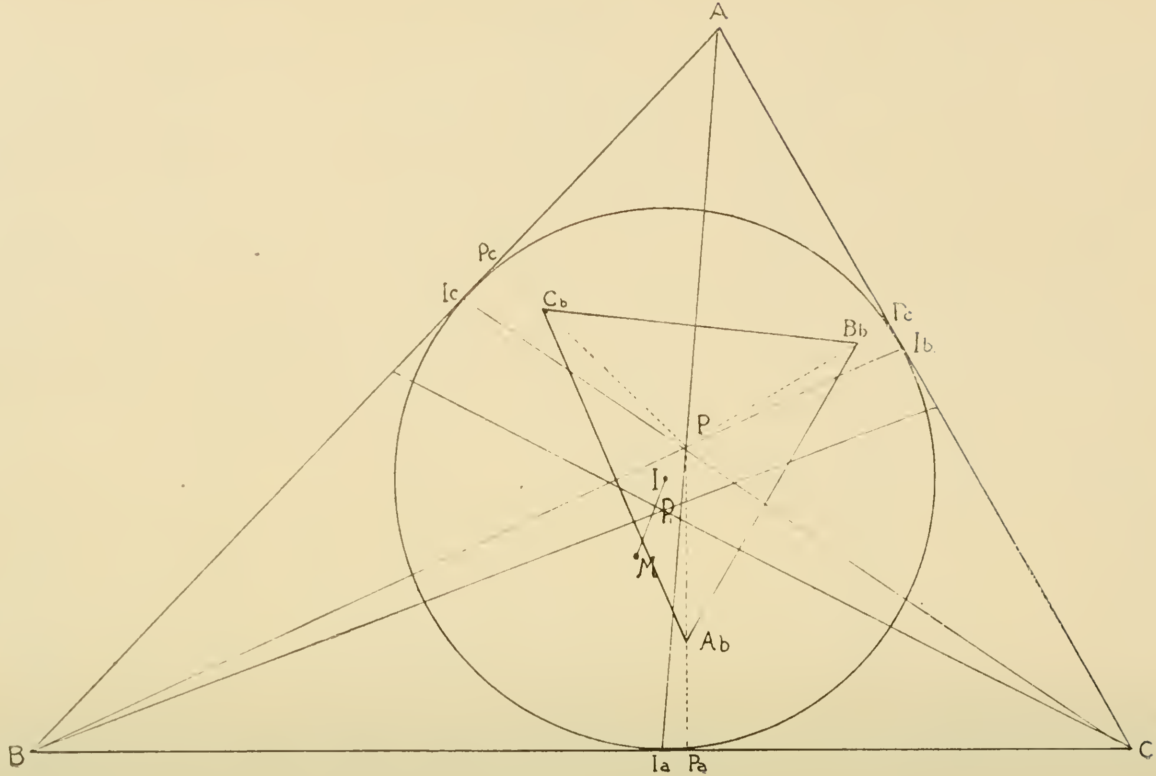
$$= \frac{8}{8S\Delta} \left\{ 8(S-a)\ (S-b)\ (S-c) - a\ (-a^2 + b^2 + c^2) \right\}$$

$$= \frac{S}{8S\Delta} \left\{ a^2\,b + a^2\,c + b^2\,c + bc^2 - 2\,abc - b^3 - c^3 \right\}$$

$$Iz : Ix = \frac{\Delta}{s(s\Sigma a^2 - \Sigma a^3)}\ (a^2b + a^2c + b^2c + bc^2 - 2abc - b^3 - c^3) :$$

$$: \frac{s}{8s\nabla}\ (a^2b + a^2c + b^2c + bc^2 - 2abc - b^3 - c^3) =$$

$$= 8\Delta^2 : s(s\Sigma a^2 - \Sigma a^3).$$


A
B
C
Pc
Ic
Cb
Bb
Γc
Ib
P
I
P₁
M
Ab
Ia
Pa

Similarly

$$IZ_2 : IX_2 = 8\Delta^2 : s(s\Sigma a^2 - \Sigma a^3)$$

And the same is true of $IZ_3 : IX_3$.

The points are therefore collinear.

$IZ_1 : IX_1 = IP_1 : IM$

$IP_1 : IM = 8\Delta^2 : S(S\Sigma a^2 - \Sigma a^3)$

$IP_1 : P_1M = IP_1 : IM - IP_1 = 8\Delta^2 : s(s\Sigma a^2 - \Sigma a^3) - 8\Delta^2 =$

$$= (-a+b+c)(a-b+c)(a+b-c) : s\Sigma a^2 - \Sigma a^3$$
$$-(-a+b+c)(a-b+c)(a+b-c).$$

The ratio of division is too complex for ordinary use.

If upon the lines PP_a, PP_b, PP_c equal distances from P be taken the triangle $A_6B_6C_6$ thus formed is similar to Nagel's triangle $A_3B_3C_3$.

For $\angle A_6PB_6 = \Pi - C$

And $\angle PA_6B_6 = \angle PB_6A_6 = \frac{1}{2}(\Pi - \angle A_6PB_6) = \frac{1}{2}C$.

Similarly the $\angle PB_6C_6 = \frac{1}{2}A$.

And hence $\angle A_6B_6C_6 = \frac{1}{2}A + \frac{1}{2}C = \frac{1}{2}(A+C)$.

Likewise $\angle B_6C_6A_6 = \frac{1}{2}(A+B)$

And $\angle B_6A_6C_6 = \frac{1}{2}(B+C)$.

But these are the angles of Nagel's triangle and therefore $A_6B_6C_6$ is similar to $A_3B_3C_3$.

P is the symmedian point of the triangle $I_aI_bI_c$. (Proc. Edinburgh Math. Soc., Vol. XI., page 105).

If through P lines are drawn parallel to the I_aI_b, I_bI_c, I_cI_a respectively, the six points of intersection with the sides are concyclic. The circle is call Adam's circle.