Application of the Cauchy Parameter Method to the Solution of Difference Equations.

By T. E. Mason.

In the application of the Cauchy parameter method to the solution of difference equations the following are the necessary steps:

1) Break the equation up into two parts, one of which gives a part $f_1(x)$ which may be readily solved and multiply the other part of the equation by the parameter t, so that the equation

$$f(x)=0$$

becomes

$$\text{(a)} \qquad f_1(x)+tf_2(x)=0.$$

2) Assume a solution of the form

$$U(x)=A(x)+B(x)t+C(x)t^2+D(x)t^3+\dots\dots\dots$$

3) Substitute in the equation (a) and equate the coefficients of the different powers of t to zero and solve. Then the parameter t is made equal to 1.

4) The solution

$$U(x)=A(x)+B(x)+C(x)+D(x)+\dots\dots\dots\dots$$

must be shown to be convergent and to satisfy the original equation.

In breaking up the equation it is necessary to make such division that the resulting solution is convergent. In equations with constant coefficients the solution of the resulting equations is, in general, no easier than the solution of the original equation, so that this method of solution is of little or no value there.

By a proper division of the equation the method of Cauchy will give the same results as the method of successive approximations. Let us illustrate this by means of the example

$$\Delta U(x)=\Phi(x)U(x),$$

where*

$$\Phi(x)=\Phi'' x^{-2}+\Phi''' x^{-3}+\dots\dots$$

*The general linear homogeneous difference equation of first order may be transformed to this form by a transformation of the form

$$g(x)=x\alpha^x a^x x^m f(x),$$

where the α, a and m are constants to be determined for the particular equation.

Let us write

$$\Delta U(x) - t\phi(x)U(x) = 0$$

and assume for the solution

$$U(x) = A(x) + B(x)t + C(x)t^2 + D(x)t^3 + \ldots\ldots$$

Then

$$\Delta U(x) = \Delta A(x) + \Delta B(x)t + \Delta C(x)t^2 + \Delta D(x)t^3 + \ldots.$$

Substituting in the equation we have

$$\Delta A(x) + t[\Delta B(x) - \phi(x)A(x)] + t^2[\Delta C(x) - \phi(x)B(x)] + t^3[\Delta D(x) - \varphi(x)C(x)] + \ldots = 0.$$

Equating the coefficients of the powers of t to zero we have

$$\Delta A(x) = 0$$

$$\Delta B(x) - \phi(x)A(x) = 0 \quad \text{or} \quad \Delta B(x) = \phi(x)A(x)$$

$$\Delta C(x) - \phi(x)B(x) = 0 \quad \text{or} \quad \Delta C(x) = \phi(x)B(x)$$

$$\Delta D(x) - \varphi(x)C(x) = 0 \quad \text{or} \quad \Delta D(x) = \phi(x)C(x)$$

$$\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots$$

Solving we have

$$A(x) = 1$$

$$B(x) = S_x\phi(x), \text{ where } S_x\phi(x) = -\sum_{i=0}^{\infty} \phi(x+i)$$

$$C(x) = S_x\phi(x)S_x\phi(x)$$

$$D(x) = S_x\phi(x)S_x\phi(x)S_x\phi(x)$$

$$\ldots\ldots\ldots\ldots$$

$$\therefore U(x) = 1 + S_x\varphi(x) + S_x\phi(x)S_x\phi(x) + S_x\phi(x)S_x\phi(x)S_x\phi(x) + \ldots\ldots$$

This series has been proven to be convergent* and gives a particular solution of the linear homogeneous equation of the first order.

But this parameter method may be applied in such a way as to obtain solutions different from those obtained by the ordinary method of successive approximations. We shall illustrate this remark by the solution of the equation

$$\Delta^2U(x) - aU(x) = x^{(n)}‡, \quad a < 1.$$

Let us write

$$\Delta^2U(x) - x^{(n)} - taU(x) = 0$$

and assume the solution

$$U(x) = A(x) + B(x)t + C(x)t^2 + D(x)t^3 + \ldots.$$

*Carmichael, Transactions American Mathemathical Society, Vol. 12, No. 1, p. 101. If in that discussion we put a=1, m=0, the two problems are identical.

‡$x^{(n)} = x(x-1)(x-2)\ldots\ldots(x-n+1)$.

Substituting in the equation and equating to zero the coefficients of the powers of t, we have

$$\begin{array}{lll} \Delta^2 A(x) - x^{(n)} = 0 & \text{or} & \Delta^2 A(x) = x^{(n)} \\ \Delta^2 B(x) - aA(x) = 0 & \text{or} & \Delta^2 B(x) = aA(x) \\ \Delta^2 C(x) - aB(x) = 0 & \text{or} & \Delta^2 C(x) = aB(x) \\ \Delta^2 D(x) - aC(x) = 0 & \text{or} & \Delta^2 D(x) = aC(x) \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \end{array}$$

$$\Delta^2 A(x) = x^{(n)}$$

$$\Delta A(x) = \frac{x^{(n+1)}}{n+1} + p_1(x)$$

$$A(x) = \frac{x^{(n+2)}}{(n+2)^{(2)}} + p_1(x)\,.\,x + p_2(x)$$

$$\Delta^2 B(x) = \frac{ax^{(n+2)}}{(n+2)^{(2)}} + ap_1(x)\,.\,x + ap_2(x)$$

$$B(x) = \frac{ax^{(n+4)}}{(n+4)^{(4)}} + ap_1(x)\frac{x^{(3)}}{3!} + ap_2(x)\frac{x^{(2)}}{2!}$$

$$\Delta^2 C(x) = \frac{a^2x^{(n+4)}}{(n+4)^{(4)}} + a^2p_1(x)\frac{x^{(3)}}{3!} + a^2p_2(x)\frac{x^{(2)}}{2!}$$

$$C(x) = \frac{a^2x^{(n+6)}}{(n+6)^{(6)}} + a^2p_1(x)\frac{x^{(5)}}{5!} + a^2p_2(x)\frac{x^{(4)}}{4!}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\therefore U(x) = \frac{x^{(n+2)}}{(n+2)^{(2)}} + \frac{ax^{(n+4)}}{(n+4)^{(4)}} + \frac{a^2x^{(n+6)}}{(n+6)^{(6)}} + \ldots\ldots + p_1(x)\left[x + \frac{ax^{(3)}}{3!} + \frac{a^2x^{(5)}}{5!} + \frac{a^3x^{(7)}}{7!} + \ldots\ldots\right] + p_2(x)\left[1 + \frac{ax^{(2)}}{2!} + \frac{a^2x^{(4)}}{4!} + \frac{a^3x^{(6)}}{6!} + \ldots\ldots\right].$$

Since $a<1$ these series converge, and it can readily be shown by substitution that this does afford a solution of the equation.

If we denote the solution of the previous equation by $U^{(n)}(x)$, then the solution of the equation

$$\Delta^2 U(x) - aU(x) = P(x), \quad a<1,$$

where P(x) is a polynomial in x of the form

$$P(x) = a_0 + a_1x^{(1)} + a_2x^{(2)} + a_3x^{(3)} + \ldots\ldots + a_mx^{(m)},$$

may be written in the form

$$U(x) = \sum_{n=0}^{m} a_nU^{(n)}(x).$$

The 2m+2 periodic functions combine into 2 independent ones.

The solution of other examples would follow the same method.

Bloomington, Ind.