On Linear Difference Equations of the First Order With Rational Coefficients.

By Thos. E. Mason.

This paper treats of the behavior of the solutions of a first order linear difference equation with rational coefficients as the variable approaches infinity in a strip parallel to the axis of imaginaries. A unique characterization of certain solutions is obtained to within the determination of a finite number of constants. The same problem has been discussed by Mellin.* The treatment here given is much shorter and simpler. The proof has been simplified by making use of the asymptotic expansion for the gamma function and by lemma II found in §1 of this paper. The use of this lemma has also permitted the removal of some restrictions made by Mellin.

Carmichael† has shown that certain solutions of the first order homogeneous linear difference equation are uniquely characterized by their behavior as the variable approaches infinity in the positive or the negative direction parallel to the axis of reals.

§1. Lemmas.

Lemma I. *If* $x = z + iz'$, $x_j = u_j + iv_j$, $x'_j = u'_i + iv'_j$, *then*

$$\lim_{z' \doteq \pm\infty} \left| \frac{\Gamma(x-x_1)\ldots\Gamma(x-x_m)}{\Gamma(x-x'_1)\ldots\Gamma(x-x'_n)} z'^{(m-n)(-z+\frac{1}{2})-k} e^{\frac{m-n}{2}\pi|z'|} e^{O+(m-n)z+k} \right| = c,$$

where

$$O_j = \tan^{-1}\frac{z'-v_j}{z-u_j},\quad O'_j = \tan^{-1}\frac{z'-v'_j}{z-u'_j} \text{ and } O = \sum_{j=1}^{n} v'_j O'_j - \sum_{j=1}^{m} v_j O_j,$$

and where‡

$$k = \sum_{j=1}^{n} R(x'_j) - \sum_{j=1}^{m} R(x_j).$$

*Acta Mathematica 15 (1891): 317-384. See §§1-3 of the paper. In §3 of an artiele in Mathematische Annalen 68 (1910): 305-337, Mellin has defined a function by means of the linear homogeneous equation

$$F(x+1)-r(x)F(x)=O,$$

where r (x) has the *particular* form

$$r(x) = \pm\frac{(x-x_1)\ldots\ldots(x-x_m)}{(x-x'_1)\ldots(x-x'_n)}$$

†Transactions of the American Mathematical Society 12 (1911): 99-134.

‡R(x) is used to denote the real part of x.

We make use of the following form of Stirling's formula:

$$\Gamma(x - x_1) = (x - x_1)^{x - x_1 - \frac{1}{2}} e^{-x + x_1} \sqrt{2\pi}\,(1 + \epsilon_x),$$

where ϵ_x approaches zero as x approaches infinity in such way that its distance from the negative axis of reals approaches infinity.

Then we have

$$\lim_{x \doteq \infty} \left| \Gamma(x - x_1) \,.\, (x - x_1)^{-x + x_1 + \frac{1}{2}} e^{x - x_1} \right| = \overline{c}.$$

Set $x = z + iz^1$ and $x_1 = u_1 + iv_1$, where z, z^1, u_1, v_1 are real, and let x approach infinity, $A \leqq R(x) \leqq A + 1$; then we have

$$\lim_{z^1 \doteq \pm\infty} \left| \Gamma(x - x_1) \,.\, (z + iz^1 - u_1 - iv_1)^{-z - iz^1 + u_1 + iv_1 + \frac{1}{2}} e^{z + iz^1 - u_1 - iv_1} \right| = \overline{c}.$$

Hence

$$\lim_{z^1 \doteq \pm\infty} \left| \Gamma(x - x_1) \,.\, e^{(-z - iz^1 + u_1 + iv_1 + \frac{1}{2})\left(\log \sqrt{(z - u_1)^2 + (z^1 - v_1)^2} + i\Theta_1\right)} e^{z - u_1} \right| = \overline{c},$$

where

$$\Theta_1 = \tan^{-1} \frac{z^1 - v_1}{z - u_1}.$$

Now $z - u_1 > 0$, therefore when $z^1 \doteq +\infty$, $\Theta_1 \doteq \frac{\pi}{2}$ and when $z^1 \doteq -\infty$, $\Theta_1 \doteq -\frac{\pi}{2}$. Thus in the above limit after mulitplying the factors in the exponent of e, we can replace $z^1\Theta_1$ by $\frac{\pi}{2}|z^1|$. Then by rearrangement and simplification we can write

$$\lim_{z^1 \doteq \pm\infty} \left| \Gamma(x - x_1) \,.\, z^{1\, -z + u_1 + \frac{1}{2}} e^{\frac{\pi}{2}|z^1|} e^{z - u_1 - \Theta_1 v_1} \right| = \overline{c}.$$

Making use of limits of this form for each of the gamma functions in the expression in the lemma, we have the lemma.

Lemma II. *If* p (x) *is a periodic function of period 1 which is analytic everywhere in the finite plane and as* $z^1 \doteq \pm\infty$ $(x = z + iz^1)$ *satisfies the relation**

$$(1) \quad \mathop{L}_{z^1 \doteq \pm\infty} \left| p(x)\, e^{-t\pi|z^1| - Qz^1} \right| = b,$$

b *finite*, t *positive, then* p(x) *may be written in the form*

$$(2) \quad p(x) = \sum_{j=-r}^{q} B_j e^{2\pi ijx}$$

*$\mathop{L}_{z^1 \doteq \pm\infty}$ denotes the greatest value approached as $z^1 \doteq \pm\infty$.

where q *is the greateat integer* $\leqq \frac{t}{2} - \frac{Q}{2\pi}$ *and* r *is the greatest integer* $\leqq \frac{t}{2} + \frac{Q}{2\pi}$; *and conversely, every periodic function of period* 1 *which can be written in the form* (2) *is analytic in the finite plane and satisfies a relation of the form* (1).

Since p(x) is periodic of period 1, it takes in any period strip all the values it takes anywhere in the finite plane. The transformation

$$w = e^{2\pi i x}$$

carries a single period strip of the x-plane into the whole w-plane, $z^1 \doteq +\infty$ corresponding to $w \doteq 0$, and $z^1 \doteq -\infty$ to $w \doteq \infty$.

We can now write

$$p(x) = f(w)$$

and since f(w) can have only the singular points zero and infinity it is expansible in a Laurent series

$$f(w) = \sum_{j=-\infty}^{\infty} B_j w^j$$

valid throughout the finite plane except at zero.

Using the fact that

$$|w| = e^{-2\pi z^1}$$

we get

$$\left| p(x)\, e^{-t\pi|z^1| - Q z^1} \right| = \left| f(w)\, w^{\frac{t}{2} + \frac{Q}{2\pi}} \right|$$

when z^1 is positive, and

$$\left| p(x)\, e^{-t\pi|z^1| - Q z^1} \right| = \left| f(w)\, w^{-\left(\frac{t}{2} - \frac{Q}{2\pi}\right)} \right|$$

when z^1 is negative. As $z^1 \doteq +\infty$, $w \doteq 0$ and

$$\operatorname*{L}_{w \doteq o} \left| f(w)\, w^{\frac{t}{2} + \frac{Q}{2\pi}} \right| = b.$$

Hence the part of the series f(w) with negative exponents can not have coefficients different from zero for j greater than the greatest integer $\leqq \frac{t}{2} + \frac{Q}{2\,\pi}$.

As $z^1 \doteq -\infty$, $w \doteq \infty$ and

$$\operatorname*{L}_{w \doteq \infty} \left| f(w)\, w^{-\left(\frac{t}{2} - \frac{Q}{2\pi}\right)} \right| = b.$$

Hence the part of the series f(w) with positive exponents can not have coefficients different from zero for j greater than the greatest integer $\leqq \frac{t}{2} - \frac{Q}{2\pi}$.

Therefore we can write

$$p(x) = \sum_{j=-r}^{q} B_j w^j = \sum_{j=-r}^{q} B_j e^{2\pi ijx},$$

where q is the greatest integer $\leqq \frac{t}{2} - \frac{Q}{2\pi}$ and r is the greatest integer $\leqq \frac{t}{2} + \frac{Q}{2\pi}$. From the definition of Q given in §2 the values of q and r will not differ by more than 1 in the problem of this paper.

The converse is obvious.

§2. Homogeneous Equations.

Theorem. *Every first order linear homogeneous difference equation with rational coefficients, as*

$$F(x+1) - r(x)\,F(x) = 0,$$

where r(x) *can be written in the form*

$$r(x) = a\frac{(x - x_1)\ldots(x - x_m)}{(x - x'_1)\ldots(x - x'_n)},\quad a = he^{iQ},\ -\pi < Q \leqq \pi,$$

has a solution F(x) *which has the following properties, provided that each of the numbers* $\frac{m-n}{4} \pm \frac{Q}{2\pi}$ *is greater than zero, or in case* m = n *that* Q = 0 *and*

$$k = \sum_{j=1}^{n} R(x'_j) - \sum_{j=1}^{m} R(x_j) \leqq 0.$$

I. F(x) *is analytic in the finite part of the x-plane defined by* R(x) > D, *where* D *is the greatest among the real parts of* $x_1, x_2, \ldots, x_m$.

II. *As* x *approaches infinity in the strip parallel to the axis of imaginaries defined by* $A \leqq R(x) \leqq A+1$ (A > D) *the absolute value of* F(x) *remains finite.*

Every such function F(x) *can be written in the form*

$$F(x) = a^x \frac{(x - x_1)\ldots(x - x_m)}{(x - x_1^1)\ldots(x - x_n^1)} \sum_{j=-r}^{q} B_j e^{2\pi ijx},$$

where q *is the greatest integer** $< \frac{m-n}{4} - \frac{Q}{2\pi}$ *and* r *the greatest interger** $< \frac{m-n}{4} + \frac{Q}{2\pi}$.

*The inequality sign should be replaced by the equality sign in case each quantity $\frac{m-n}{4} \mp \frac{Q}{2\pi}$ is an interger and at the same time the exponent of z^1 in the expression in lemma I, §1, is $\geqq 0$, that is when $(m-n)(-z+1/2) - k \geqq 0$ for all values of x in the strip define l in condition II of the theorem.

The quantity $a^{x}\bar{F}(x)$ evidently satisfies the difference equation of the theorem, where

$$\bar{F}(x) = \frac{\Gamma(x - x_1) \dots \Gamma(x - x_m)}{\Gamma(x - x_1^1) \dots \Gamma(x - x_n^1)}.$$

$a^{x}\bar{F}(x)$ also satisfies I since in the region defined the gamma functions in the numerator are analytic and in the denominator are different from zero. $a^{x}\bar{F}(x)$ being a particular solution of the difference equation, the general solution is

$$F(x) = p(x)a^{x}\bar{F}(x),$$

where $p(x)$ is an arbitrary periodic function of period 1.

From the limit in Lemma I, §1, it is evident that I and II will be satisfied if, and only if, $p(x)$ is chosen so that it is analytic everywhere in the finite plane and when $x \doteq \infty$, $A \leqq R(x) \leqq A + 1$, satisfies the relation

$$\underset{z^1 \doteq \pm\infty}{L} \left| \frac{a^{x} p(x)}{z^{1\,(m-n)(-z+\frac{1}{2})-k}\; e^{O+(m-n)z+k}\; e^{\frac{m-n}{2}\pi|z^1|}} \right| = b,$$

where b is finite. This can be written $\left(|a^{x}| = h^{z} e^{-Qz^1}\right)$

$$\underset{z^1 \doteq \pm\infty}{L} \left| \frac{p(x)\, e^{-\left(\frac{m-n}{2}\right)\pi|z^1| - Qz^1}\, h^{z}}{z^{1\,(m-n)(-z+\frac{1}{2})-k}\; e^{O+(m-n)\,z+k}} \right| = b.$$

The use of Lemma II, §1, gives the form which $p(x)$ must take to satisfy this relation and thus completes the proof of the theorem.

$F(x)$ will in general be uniquely determined if its value is known at $q+r+1$ different points at which it is analytic. For then we should have a set of $q+r+1$ equations linear in the B's from which we could determine the constants B_j.

The form of the periodic function $p(x)$ obtained by Mellin is

$$p(x) = \sin\pi(x - c_1) \dots\dots \sin\pi(x - c_p) \left[\frac{A_1}{\sin\pi(x - c_1)} + \dots + \frac{A_p}{\sin\pi(x - c_p)} \right],$$

where the c's are arbitrary with the exception that no two can differ by an integer. Mellin restricted a to be a real positive quantity and in case this is done the q and r of this paper become equal. In that case the identity of the periodic function of Mellin and the periodic function of this paper can be shown by making the transformation $w = e^{2\pi ix}$ and equating the coefficients of like powers of w in the two transformed expressions for p(x). This gives a system of p linear equations to determine $A_1, A_2, \ldots, A_p$, where the p of Mellin's paper is equal $2q+1$.

§3. Non-homogeneous Equations.

Theorem. *If* r(x) *and* s(x) *are rational functions of the form**

$$r(x) = a\frac{(x-x_1)\ldots(x-x_m)}{(x-x^1_1)\ldots(x-x^1_n)},\quad s(x) = b\frac{(x-x_1^{11})\ldots(x-x^{11}_g)}{(x-x^1_1)\ldots(x-x^1_n)},$$

where $m \geqq n$, *then the series*

$$S(x) = \sum_{t=o}^{\infty}\frac{s(x+t)}{r(x+t)\,r(x+t-1)\ldots r(x)}$$

is always uniformly convergent for $|a|>1$ *and for* $|a|=1$ *when* $m>n$, *and is uniformly convergent for* $|a|=1$, $m=n$, *when* $k-(g-n)>1$, *where*

$$k = \sum_{j=1}^{n} R(x_j^1) - \sum_{j=1}^{m} R(x_j).$$

If the conditions for the uniform convergence of S(x) *are fulfilled, then every first order linear non-homogeneous difference equation with rational coefficients, as*

$$F(x+1) - r(x)F(x) = s(x),$$

has a solution F(x) *which has the following properties:*

I. F(x) *is analytic in the part of the finite x-plane defined by* $R(x)>D$, *where* D *is the greatest among the real parts of* $x_1, x_2, \ldots, x_m$.

II. *If* x *is confined to the strip parallel to the axis of imaginaries defined by* $A \leqq R(x) \leqq A+1$ $(A>D)$ *the absolute value of* F(x) *remains finite as* x *approaches infinity.*

Every such solution F(x) *can be written in the form*

$$F(x) = a^x\frac{\Gamma(x-x_1)\ldots\ldots\Gamma(x-x_m)}{\Gamma(x-x^1_1)\ldots\ldots\Gamma(x-x^1_n)}\sum_{j=r}^{q} B_j e^{2\pi ijx} - \sum_{t=o}^{\infty}\frac{s(x+t)}{r(x+t)r(x+t-1)\ldots r(x)},$$

r *and* q *being defined as in the theorem in* §2.

In the equation

$$F(x+1) - r(x)F(x) = s(x)$$

make the substitution

$$F(x) = f(x)u(x),$$

where f(x) is the solution of the homogeneous equation given in the theorem

*If r(x) and s(x) do not already have a common denominator they can easily be reduced to expressions with a common denominator.

in §1. This gives

$$f(x+1)u(x+1) - r(x)f(x)u(x) = s(x).$$

Since $f(x+1) - r(x)f(x) = 0$ we can divide by $r(x)f(x)$ and we have

$$u(x+1) - u(x) = \frac{s(x)}{r(x)f(x)}$$

or

$$u(x) = p(x) - \sum_{t=0}^{\infty} \frac{s(x+t)}{r(x+t)f(x+t)}$$

where $p(x)$ is an arbitrary periodic function of period 1. Now

$f(x+t) = r(x+t-1)f(x+t-1) = \ldots\ldots\ldots = r(x+t-1)r(x+t-2) \ldots\ldots r(x)f(x)$.

Making this substitution in the preceding equation we have

$$u(x) = p(x) - \sum_{t=0}^{\infty} \frac{s(x+t)}{r(x+t)\, r(x+t-1) \ldots\ldots r(x)f(x)}.$$

If we choose $p(x) = 0$ we have

$$F(x) = f(x)u(x) = -\sum_{t=0}^{\infty} \frac{s(x+t)}{r(x+t)\, r(x+t-1) \ldots\ldots r(x)}.$$

$F(x)$ is then a solution satisfying I and II provided that

$$S(x) \equiv u_0(x) + u_1(x) + u_2(x) + \ldots\ldots, \qquad u_n(x) = \frac{s(x+n)}{r(x+n)\, r(x+n-1) \ldots r(x)},$$

is analytic. $S(x)$ is analytic provided that it converges uniformly in any closed region T lying in the strip defined by the relation $A \leqq R(x) \leqq A+1$.

In the region T in the strip under consideration the following ratio of the $(t+1)$th term to the t-th term holds for every value of x in that region.

$$\left|\frac{u_{t+1}}{u_t}\right| = \left|\frac{1}{r(x+t)} \frac{s(x+t)}{s(x+t-1)}\right|$$

(5) $$= \left|\frac{1}{a}\left\{t^{n-m} - [k - \overline{n-m}\, R(x)]t^{n-m-1} + \ldots\ldots\right\} \left\{1 + \frac{l}{t} + \frac{l_1(x)}{t^2} + \ldots\right\}\right|,$$

where k has the same meaning as in §1 and $l = g - n$. When $n = m$ (5) becomes

(6) $$\left|\frac{u_{t+1}}{u_t}\right| = \frac{1}{|a|}\left|\left\{1 + \frac{-k+l}{t} + \ldots \text{ terms in } \frac{1}{t^2}, \frac{1}{t^3}, \text{ etc.}\right\}\right|.$$

In considering the value of this ratio we shall need to examine the following cases:

(1). When $n>m$ (5) shows the ratio to be greater than 1 and therefore the series $S(x)$ diverges.

(2). When $n<m$ (5) shows that for increasing t the ratio approaches zero and therefore the series $S(x)$ converges.

(3). When $n=m$ we see from (6) that the convergence of the series depends on the value of $|a|$.

If $|a|>1$ the ratio ultimately approaches a quantity less than 1 and therefore $S(x)$ converges.

If $|a|<1$ the ratio is greater than 1 and $S(x)$ diverges.

If $|a|=1$ the series will converge when* $k-l>1$.

In the cases where $S(x)$ converges, except where $n=m$ and $|a|=1$, the ratio u_{t+1}/u_t has been shown to approach a quantity which is less than 1 for every x in T. Hence an M and an r can be found such that

$$M+Mr+Mr^2+Mr^3+Mr^4+\dots\dots\dots$$

is a convergent series of positive constant terms which is greater term by term than the series

$$(7) \qquad u_0+u_1+u_2+u_3+\dots\dots$$

for every x in T. Therefore the series $S(x)$ converges uniformly in T and is an analytic function in that region since each term is analytic in T. In case $n=m$ and $|a|=1$ we see from (6) that the coefficient of $1/t$ does not contain x but that the coefficients of higher powers of $1/t$ do. These coefficients are polynomials in x. If we replace each x by a quantity which is greater than the greatest absolute value of x in T and replace the coefficients of the powers of x by their absolute values, then the ratio (6) is increased but is still such that a series of positive constants can be constructed which is convergent and is term by term greater than the series (7). Hence $S(x)$ converges uniformly in T when $n=m$ and $|a|=1$ and is therefore analytic in T. But T is any closed region in the strip and hence $S(x)$ is analytic throughout the strip.

Under the conditions of the theorem $S(x)$ has been shown to be a solution of the difference equation of the theorem with the required properties I and II. The general solution having those properties will be obtained by adding to this particular solution the general solution of the homogeneous equation as found in the theorem of §2 which has the same properties. This completes the theorem.

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*In accordance with a theorem of Gauss. See Opera, vol. 3, p. 139.

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