Conjugate Functions and Canonical Transformations.

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(Abstract.)

It is known that any function, $\phi(Z)$, of a complex variable, $Z = x + iy$, may be separated into a real part $\phi_1(x, y)$ an imaginary part, $i\varphi_2(x, y)$, and that φ_1, φ_2 each satisfy Laplace's equation $\frac{\delta^2 \phi}{\delta x_2} + \frac{\delta^2 \phi}{\delta y_2} = o$ * A very elegant geometric interpretation of these two functions φ_1, ϕ_2 may be had by equating each to a third variable ζ : $\phi_1(x, y) = \zeta$, $\phi_2(x, y) = \zeta$. Each equation then represents a surface for any point of which Laplace's equation is true. By developing $\zeta = \phi_1(x, y)$ into a power series in the vicinity of any point x_o, y_o, and using the Laplace equation, we have the theorem: the projection of the section of a tangent plane to the surface $\zeta = \phi_1(x, y)$ upon the x, y-plane is a curve having a double point at x_o, y_o with real, orthogonal tangents, and hence the surface is hyperbolic at every point.

$\zeta = k$ gives lines of level on $\zeta = \phi_1(x, y)$, while $\zeta = k_2$ in $\zeta = \phi_2(x, y)$ gives cylinders which intersect $\zeta = \phi_1(x, y)$ in *curves of quickest descent.*

The second part of the paper deals with the linear fractional function $Z_1 = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}$ which has the fundamental invariant points f_1, f_2 about which a canonical transformation may be constructed so that $Z = o$, when $Z^1 = f_1$; $Z = \infty$, $Z' = f_2$. This function is $Z = \frac{Z^1 - f_1}{Z^1 - f_2} = \frac{\alpha - \gamma f_1}{\alpha - \gamma f_2}\left(\frac{Z - f_1}{Z - f_2}\right)$. The modulus of $\frac{Z - f_1}{Z - f_2}$, and amplitude of $\frac{Z - f_1}{Z - f_2}$, set, respectively, equal to constants give an elliptic system and an hyperbolic system of circles about and through the two points f_1, f_2. Now the transformation

$$Z = \frac{Z^1 - f_1}{Z_2 - f_2} = \frac{\alpha - \gamma f_1}{\alpha - \gamma f_2}\left(\frac{Z - f_1}{Z - f_2}\right),$$

sets up a motion about f_1, f_2 which is determined by the modulus and the amplitude of $\frac{\alpha - \gamma f_1}{\alpha - \gamma f_2}$. If mod. ∓ 1 and amp $= o$, motion

*Where $\frac{\delta^2 \Phi}{\delta x^2}$ denotes the second partial of Φ with regard to x, and so for $\frac{\delta^2 \Phi}{\delta y^2}$.

goes along the hyperbolic circles, the elliptic circles interchanging. If mod. $= 1$, amp. $\mp o$, motion goes along elliptic circles, the hyperbolic system being invariant. If mod. ± 1, amp. $\mp o$, motion is along neither family but passes diagonally from curvilinear rectangle to curvilinear rectangle. These respective transformations may be named *hyperbolic*, *elliptic*, *loxodromic*. The circles about and through the fundamental points are potential lines and lines of flow in the well known problem of electricity of equal source and sink.

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