

## ON THE POINCARÉ TRANSFORMATION.

TOBIAS DANTZIG.

1. *Introductory.*

In a memoir entitled "Sur un Theoreme de Geometrie" (Rendiconti del Circolo Matematico di Palermo, Vol. 33, 1912, P. 375-407) the late Henri Poincaré has considered a certain type of transformations of fundamental value in Celestial Mechanics. Without giving a proof he has announced there a general property of all such transformations. The proposition has since been taken up by George D. Birkhoff who in his paper "Proof of Poincaré's Geometric theorem" Transaction of the American Mathematical Society, Vol. 14, 1913) has given the theorem a general demonstration. His proof lacks, however, simplicity and directness.

In my article entitled "Demonstration directe du dernier theoreme de Henri Poincaré" which appeared in the February issue of the "Bulletin des Sciences Mathematiques et Astronomiques." I gave an elementary, genetic proof of the proposition. I wish to reproduce here the main features of my demonstration as well as to bring out in greater detail some points which were left incomplete in the said paper.

2. *Poincaré's Theorem.*

Slightly generalized\* the theorem can be stated thus:

Let  $T$  be a transformation operating in a plane and having the following properties:

(a) It is continuous and one-to-one in the ring formed by two closed curves contours ( $C$ ) and ( $c$ ) of which ( $c$ ) is entirely within ( $C$ ). (Fig. 1.)

(b) It leaves the two contours ( $C$ ) and ( $c$ ) invariant.

(c) It moves any point  $M$  on ( $C$ ) into a point  $M$  in the positive sense of rotation, while the points  $m$  on ( $c$ ) advance in the opposite sense.

(d) It takes every point  $P$  within the ring ( $Cc$ ) into a point  $P$  also within the ring.

(e) It conserves arcs.

Under these considerations there are within the ring ( $Cc$ ) at least two points  $I$  and  $J$  which are left invariant by  $T$ .

3. *Notations.*

Choose at random within ( $c$ ) (Fig. 1) a point  $O$ , and a half-line  $OX$ , for pole and polar axis respectively, and let

$$(1) \begin{cases} \bar{r} = f(r, \theta) \\ \bar{\theta} = g(r, \theta) \end{cases}$$

be the polar equations of the transformation.  $r, \theta; \bar{r}, \bar{\theta}$  are the co-ordinates

of any point  $P$  and its image  $\bar{P}$ , while  $f$  and  $g$  are functions which by hypothesis are continuous and single valued within the ring  $(Cc)$  and on its boundaries. The same is true of the quantity

$$(2) \quad Z = \bar{\theta} - \theta = (r, \theta)$$

which measures in value and sign the angle  $PO\bar{P}$ . I shall call  $Z$  the *deviation* for the point  $P$ . The following are properties of this function which immediately follow from the hypothesis.

*The deviation is positive for any point of the inner contour (c), negative for any point of the outer contour. (Hypothesis c.)*

*On any ray  $OM$  there exists at least one point  $D$  for which the deviation is zero. Such a point is shifted by the transformation radially only i. e.  $D$  and  $\bar{D}$  are collinear with  $O$ .*

#### 4. The locus of zero deviation.

The locus of all points  $D$  within the ring for which the deviation vanishes has for equation

$$(3) \quad z = (r, \theta) = 0$$

I shall denote this locus by  $(D)$ . The transformation exercises on this locus a central effect shifting every point  $D$  on it along the ray  $OD$ . It follows, therefore, that

*If  $E$  is a multiple point of order  $p$  on  $(D)$ ,  $\bar{E}$  is a multiple point of the same order on  $(\bar{D})$ , and  $E$  and  $\bar{E}$  are collinear with  $O$ .*

*If a ray  $l$  touches  $(D)$  in  $A$  it will also touch  $(\bar{D})$  in  $\bar{A}$ , and the contact is of the same order.*

*If  $(D)$  possesses within  $(Cc)$  a closed branch  $(u)$  enclosed between two rays  $l$  and  $l^1$  the image  $(\bar{u})$  is also closed and is contained in the same angle.*

All these properties are immediate consequences of the hypothesis and definitions.

#### 5. The Principal Branch.

*Lemma A. The locus of zero deviation has within the ring  $(Cc)$  at least one closed branch  $(d)$  completely surrounding the inner boundary.*

Indeed, if we regard (3) as the equation in semipolar co-ordinates of a surface  $S$ , the cylinders parallel to  $Oz$  and built on  $(C)$  and  $(c)$ , will meet  $S$  in two curves  $(\Gamma)$  and  $(\gamma)$  of which  $\Gamma$  is entirely below the plane  $\Pi$  while  $\gamma$  is entirely above. The portion of the surface contained between the two cylinders is continuous and single sheeted.  $S$  therefore, will be cut by  $\Pi$  in at least one closed branch completely surrounding  $(c)$ . But the complete section of  $S$  by  $\Pi$  is the locus  $(D)$ , which proves the lemma.

The branch  $(d)$  may have multiple points, but if  $(E)$  be such a loop on  $(d)$ , the image  $(\bar{d})$  will possess a similar loop  $(\bar{E})$ . The elimination of loops on  $(d)$  will have, therefore, the effect of eliminating the loops on  $(\bar{d})$ . It is, therefore, legitimate to assume that  $(d)$ , is a simple contour, as well as its image  $(\bar{d})$ .

I shall call the curve (d) deprived of all loops the *principal branch* of the locus (D). If (D) possesses more than one such branch, the one "closest" to the inner boundary may be selected for the principal branch.

6. *A Particular Case*

I will say that a closed contour (K) is *everywhere convex* if any ray thru O meets it in one and only one point (Fig. 1). If a contour (K) is not everywhere convex, it is clear that there exist rays which touch it. By drawing all these tangent rays it is possible to divide the contour into "convex" and "concave" arcs and there is a finite number of these arcs. (Fig. 2).

It is evident from the foregoing considerations that if the principal branch is everywhere convex, this is also true of its image ( $\bar{d}$ ). In the general case by drawing the tangent rays we simultaneously divide both (d) and ( $\bar{d}$ ) into convex and concave arcs.

These preliminaries being established, the proof of the theorem is immediate in the case when the principal branch of the zero deviation curve is everywhere convex. Indeed (d) and ( $\bar{d}$ ) must in this case have at least two real intersections, for otherwise d would be either entirely within ( $\bar{d}$ ) or entirely without. In either case, the area of the ring (d, c) could not equal that of the ring ( $\bar{d}$ , c) contrary to the hypothesis of conservation of areas. If now I is a point common to (d) and ( $\bar{d}$ ), its image  $\bar{I}$  coincides with I, and the proposition is proved.

The method used here to prove that (d) and ( $\bar{d}$ ) intersect in at least two points, applies to the general case and discloses this fundamental fact: *If the point I is situated on a convex arc of the principal branch it is certainly an invariant point.* If, however, the point I is on a concave arc it may not be an invariant point, as for instance the point C in Fig. 2. The problem, therefore, reduces to showing that *at least one convex arc of the branch (d) meets its image*

7. *The Auxiliary Contour.*

I shall call an arc of zero deviation a *normal arc* if it is possible to go from one extremity of the arc to the other without changing the sense of rotation. A segment of a ray thru O is *normal* if it is possible to go from one extremity to the other without changing the sign of the deviation. A contour consisting of normal arcs and segments, I shall call a *normal contour*.

*Lemma B.* *It is always possible to construct within the ring (Cc) a closed normal contour (K) completely surrounding the boundary (c) and everywhere convex.*

I commence by drawing all the rays tangent to the zero deviation curve both in its principal and secondary branches. The locus (D) as well as its image ( $\bar{D}$ ) is thus divided in a certain number of convex and concave arcs (Figs. 2 and 3). Any one of these tangent rays  $I_1$  touches (D) in  $A_1$  and crosses it besides in a number of points  $B_1, B'_1, \dots$ . Let  $\bar{a}_1 = B_1A_1$  be a normal

are of the principal branch the rotation being in the negative sense. Take for second "leg" of the normal contour the segment  $A_2B_2 = S_2$  directed inward and in which  $B_2$  is the first point of zero deviation encountered.  $B_2$  may be a point on the principal branch (Fig. 2) or on the secondary branch (Fig. 3). Selecting then for third leg the normal arc  $a_2 = B_2A_3$  and continuing in this manner we shall finish by returning to the point  $B_1$ , after having described a closed contour ( $K$ ) everywhere convex and consisting of normal arcs and segments only. This contour is shown in the figures by the heavy lines; its image by heavy dotted lines.

### §. Proof of Poincaré's Theorem.

If  $\bar{a}_1$  is the image of the normal arc  $a_1$  it is clear that  $\bar{a}_1$  cannot intersect ( $K$ ) in any other part of it but the corresponding arc  $a_1$ , for  $a_1$  and  $\bar{a}_1$  are contained between the same two rays  $l_1$  and  $l_1 = 1$ . On the other hand if  $\bar{S}_K$  is the image of the segment  $S_K$ , then  $\bar{S}_K$  will have no other points in common with the contour ( $K$ ) than the point  $\bar{D}_K$ .

From these remarks the proof of the theorem follows without difficulty. For if we assume that there are no invariant points, no arc  $\bar{a}_1$  would have any points in common with the corresponding arc  $a_1$ . The contour ( $K$ ) would, therefore, be either entirely within or entirely without its image ( $\bar{K}$ ) and in either case the area of the ring ( $Kc$ ) could not equal that of ring ( $\bar{K}c$ ) contrary to the hypothesis of conservation of areas.

---

\*In the above mentioned article Poincaré states the theorem in the case of concentric circles. Birkhoff also considers this case, although he remarks at the end of his article that the theorem could be extended to the case of any two convex contours with the aid of a conformal transformation. This has never been very clear to me.

