Lines on the Pseudosphere and the Syntractrix of Revolution.

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Introduction.

Consider two surfaces of revolution S and S_1 generated by the revolution of the curves C and C_1 about the Z axis. C_1 is formed by taking on the tangents to C distances equal to the constant-k' times the length of the tangents. The length in each case is measured from the z-intercept toward the point of tangency. Let $C = O$ be given by $z = f(u)$, then $C_1 = O$ will be given by,

$$z_1 = (L - 1)u_1 f'(Lu_1) + f(Lu_1)$$

where $L = 1\ k'$ and the equations of transformation from S to S_1 are,

$$\begin{aligned} u &= Lu_1 \\ v &= v_1 \end{aligned} \qquad \ldots\ldots(1)$$

When the length of the tangent to the curve C is constant, as in the tractrix, the curve C_1 is the syntractrix (see Note), and the surfaces S and S_1 are therefore the pseudosphere and the syntractrix of revolution.

What follows is the study of lines on these surfaces. The geodesic lines on the pseudosphere have been studied by means of lines in the plane. This surface being one of constant negative curvature (-1) may, according to Beltrami (see Note 2), be represented geodesically by a system of straight lines in the plane.

Much of the work outlined here for geodesics on the pseudosphere may be found in Darboux, Theorie des Surfaces, Vol. III, and is given here only in the way of review and for completeness.

The claim made for the originality in this part of the work is in (1) the classification of the geodesic lines and the study of certain systems of geodesic lines and their corresponding lines in the plane; (2) *the transformations of the system of circles* into straight lines by making use of the sphere,

Note 1.—The syntractrix is defined as the curve generated by taking a constant distance on the tangents to the tractrix. Peacock, p. 175.

Note 2.—Beltrami, Annali di Matematica. Vol. 7, p. 185
Bianchi, Lukat, Differential—Geometrie, p. 436.

as indicated; (3) the study of the asymptotic lines and the loxodromic lines on the pseudosphere and their representations in the plane.

In the second part of the work the lines on the syntractrix of revolution are studied. This work so far as I know has never been done before In it I have worked out the equations of the geodesic, asymptotic and loxodromic lines. These have been studied in particular by classifying the surfaces S_1 according as $d = 2C$, where C is the length of the tangent to the tractrix and d the constant distance taken on that tangent. When $d = 2C$ it happens that the geodesic lines on S_1 are all real and that the geodesic lines for $d\ \ 2O$ are real or imaginary according as $r^2\ \ \left| k_1\ k \right|$.

The loxodromic lines are represented in the plane by the same system of straight lines as the loxodromic lines of the pseudosphere. The drawings are given for the sake of clearness.

CHAPTER I.

Geodesic Lines on the Pseudosphere.

Taking the equation of the tractrix in the form,

$x = C \cosh.^{-1} c\ y - (C^2 - y^2)^{1\ 2}$ we get for the given surface,

$$x = u \cos v \qquad \ldots\ldots(2)$$

$$y = u \sin v$$

$$z = C \cosh.^{-1} c\ u - (C^2 - u^2)^{1\ 2}$$

and the fundamental quantities of the Gaussian (see Note 1) notation are, $E = C^2\ u^2$, $F = 0$, $G = u^2$, $D = (C^2)\ (u(C^2 - u^2(^{1\ 2})$, $D' = 0$, $D'' = -u(C^2 - u^2)^{1\ 2}$, $K = -1$.

Using the method of calculus of variations as developed by Weierstruss (see Note 2) to obtain the equations of the geodesic lines, we have to minimize the integral,

$$I = \int_{t_0}^{t} (Edu^2 - 2Fdudv + Gdv^2)^{1\ 2} dt$$

$$= \int_{t_0}^{t} ((C^2 u'^2)\ (u^2) + u^2 v'^2)^{1\ 2} dt = \int_{t_0}^{t} F' dt$$

Legendre's condition for a minimum is $\bar{F}v - (d\ dv)\bar{F}v' = 0$ where $(\bar{F}v) = (\delta\bar{F})\ (\delta v)$ and $\bar{F}v' = (\delta F)\ (\delta v')$.

Here $\bar{F}v = 0$, so that we get as the equations of the geodesics

$$\bar{F}v' = (u^2 v')\ ((C^2 u^2\ u^2) + u^2 v'^2)^{1\ 2} = \alpha \qquad \ldots\ldots(3)$$

Where α is the constant of integration.

Note 1.—Bianchi, Differential-Geometrie, pp. 61 and 87.

Note 2.—Kneser, Variationsrechnung.

Osgood, Annals of Mathematics, Vol. 2, p. 105.

In considering these curves two cases may arise, (1) when $\alpha = 0$, (2) $\alpha \neq 0$. Case (1) when $\alpha = 0$, either $u = 0$ or $v' = 0$. But $u \neq 0$ hence $v' = 0$ and so $v =$ constant. That is the meridians are geodesics.

Case (2) when $\alpha \neq 0$, (3) becomes

$$v = (C/\alpha u)(u^2 - \alpha^2)^{1/2} + \beta \qquad \ldots\ldots(4)$$

This may, however, be put in a more convenient form, since in the present case the geodesic lines $v =$ constant all meet in a point and the curves $u =$ constant form a system of geodesic circles — the orthogonal trajectories of the meridians. Under such conditions E may be equated to unity (see Note 1). The new u_2 is then given by the relation $u_2 = \int (E)^{1/2} du$. Hence $u = e^{u_2/c}$. Replacing in (4) u by its value just found the equation of the geodesic lines becomes

$$v = (C/\alpha)(1 - \alpha^2 e^{-2u/c})^{1/2} + \beta \qquad \text{(see Note 2)} \ldots\ldots(5)$$

This equation may be used to determine the allowable values of α and β. The constant β being additive has no effect except to turn the surface about the z axis. Thus a geodesic line given by one value of β may be made to coincide with one given by another value of β by revolution about the z axis, α remaining constant. β may vary from $-\infty$ to $+\infty$.

From (5) it is seen that the lines are real or imaginary according as

$$\alpha^2 e^{-2u/c} \gtreqless 1,$$

(1) Let $\alpha^2 e^{-2u/c} > 1$, then $|\alpha| > e^{u/c}$.

But for the pseudosphere $u_2 > C \log C$ so that the geodesics will be imaginary when $|\alpha| > C$. (2 & 3). Let $\alpha^2 e^{-2u/c} \leqq 1$, then $|\alpha| \leqq e^{u/c}$.

Hence $|\alpha| \leqq C$ gives real geodesics.

Equations (5) may be transformed into

$\alpha^2(v^2 + C^2 e^{-2u/c}) - 2\beta\alpha^2 v + (\beta^2 \alpha^2 - C^2) = 0$ which when

$$v^2 \pm C^2 e^{-2u/c} = y$$
$$v = x \qquad \ldots\ldots(6)$$

may be represented in the plane by the straight lines,

$$y = 2\beta x - (\beta^2 - C^2/\alpha^2) \qquad \ldots\ldots(7)$$

(6) may be broken up into two transformations

(a)
$$\left.\begin{aligned} v &= x \\ Ce^{-u/c} &= y \end{aligned}\right\} \ldots\ldots(8)$$

NOTE 1.—Knoblauch, Theorie der Krummen Flächen, p. 133.

NOTE 2.—Bianchi, p. 419.

which transforms S conformally on the plane so that the geodesics lines go over into the circles,

$$(x-\beta)^2 + y^2 = C^2/\alpha^2 \qquad \text{(See Note 1)}$$

and (b)

$$\left.\begin{aligned} y &= (\bar{y} - x^2)^{1/2} \\ x &= \bar{x} \end{aligned}\right\} \ldots\ldots(9)$$

which changes the circles into the straight lines,

$$y = 2\beta x - \beta^2 + C^2/\bar{\alpha}^2 \qquad \ldots\ldots(10)$$

By (9) the x axis goes into the parabola $x^2 = y$ and all the lines $y =$ constant go into the parabolas $x^2 = y +$ constant. The whole upper part of the plane is represented inside the parabola $x^2 = y$. The points on the lines $x =$ constant are moved along the lines. The origin is the fixed point of transformation.

Circles concentric at the origin correspond to lines $y =$ constant while every system of concentric circles on the x axis goes over into a system of parallel lines. A system of circles given by (8) passing through a point corresponds to a system of lines through a point. A system of circles with the y axis as radical axis

$$x^2 + y^2 - 2\beta x + k^2 = 0$$

and their orthogonal trajectories,

$$x^2 + y^2 - 2hy = \pm d^2 \qquad \text{(See Note 2)}$$

corresponds to a sheaf of lines and a sheaf of conics.

The geodesics $v =$ constant correspond to the lines $x =$ constant i. e. to the diameters of the parabola $x^2 = y$. The entire real part of the surface S is represented in the xy—plane by the strip $y = C$ $y = C/e$ and in the xy—plane by the strip included by the curves $x^2 = y - C^2$ and $x^2 = y - C^2 e^2$. The circles of (8) tangent to the line $y = C$ e go over into a system of straight lines enveloping the parabola $x^2 = y - C^2/e^2$.

Since the representation given by (8) is conformal it is interesting to note that the lines $y =$ constant may be considered as the envelop of a system of circles of constant radii and centers on the x axis given by the equation,

$$(x-\beta)^2 + y^2 = C^2/k^2$$

corresponding on the surface to the geodesics,

$$v^2 + C^2e^{-2u/c} - 2\beta v + (\beta^2 - C^2/k^2) = 0 \qquad 0 < k \leqq e$$

NOTE 1.—Bianchi, p. 419.

NOTE 2.—Salmon's Conic Sections, p. 100.

These may be regarded as a system of geodesics having as an envelop the geodesic circles $u = k_1 \; 0 < k_1 \leqq C$. A system of concentric circles with the centers at any point $(e, 0)$ on ox gives the geodesics

$$v^2 + C^2 e^{-2u/c} - 2ev + e^2 - C^2/\alpha^2 = 0$$

If $\alpha . \beta = C$ we get a system of circles through the origin

$$x^2 + y^2 - 2\beta x = 0$$

which correspond to a system of geodesics through a point. In this case, however, the point is not a real point of S.

A system of circles with the centers on ox and passing through a point on the line $y = k$, $C\,e<k<C$ envelops a unicursal quartic of the form,

$$Ay^2 + A_1 x^2 + A_2 x^2 y^2 + 2A_3 x^2 y + 2A_4 x y^2 + 2A_5 xy = 0$$

This system of circles corresponds to a system of geodesics through a real point and the quartic curve to the geodesic envelop

$$e^{-2u/c}(A + A_2 v^2 + 2A_4 v) + e^{-u/c}(2A_3 C^{-1} v^2 + 2A_5 C^{-1} v) + (A_1 C^2) v^2 = 0$$

In this case the circles have a second common point on the line $y = -k$ so that the quartic envelope (which in this case is imaginary), having four nodes, breaks up into two circles which are themselves curves of the system and therefore correspond to the geodesics of the surface.

The orthogonal systems of circles,

$$x^2 + y^2 - 2\beta x \mp b^2 = 0$$

$$x^2 + (y - h)^2 = h^2 \pm b^2$$

having the radical axis correspond to the geodesics

$$v^2 + C^2 e^{-2u/c} - 2\beta v \pm b^2 = 0$$

and their orthogonal geodesic circles

$$v^2 + C^2 e^{-2u/c} - 2hCe^{-u/c} \pm b^2 = 0$$

These may be such that the limiting points of the circles are real and distinct, coincident or imaginary. It is interesting to note that this system of circles, which in so many problems in applied mathematics represents lines of *flow* and equipotential lines may be mapped conformally on the pseudosphere in such a way that the lines of flow and the equipotential lines are the geodesics of a system and their orthogonal geodesic circles.

Another straight line representation of the geodesic lines of the surface S.

[10—18192]

If we project stereographically upon the sphere

$$\xi^2 + \eta^2 + (\zeta - 1/2)^2 = 1/4$$

whose south pole is the point (0, 0, 0) and whose north pole is the point (0, 0, 1), the circles given by the transformation $v = x$, $Ce^{-u/c} = y$ we shall have the upper part of the xy—plane represented conformally upon the hemisphere Lbd — C. The x—axis goes into the great circle Lbd and the

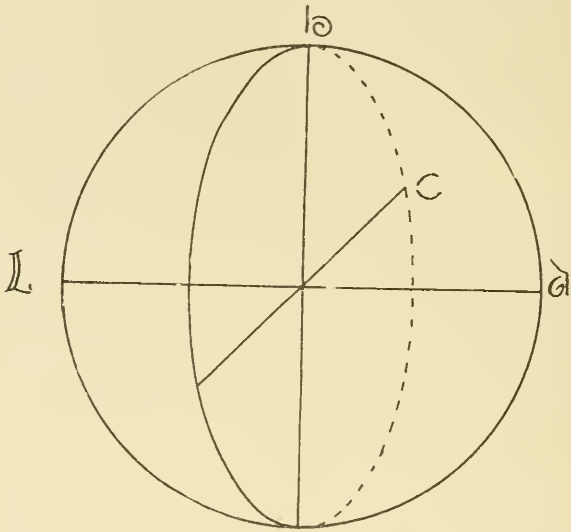

Fig. 1.

circles at right angles to o—x go into circles at right angles to Lbd.

If now we project orthogonally upon the plane Lbd we shall have the representation in question as chords of Lbd. Since $\xi\ \eta\ \zeta$ are the co-ordinates of the sphere we get as the equations of transformation from the plane to the sphere,

$$x = (\xi)/(1 - \zeta)$$
$$y = (\eta)/(1 - \zeta)$$

This gives for the circle

$$x^2 + y^2 - 2\beta x + \beta^2 - C^2\alpha^2 = 0$$

the plane

$$(1 - \beta^2 + C^2\alpha^2)\zeta - 2\beta\xi + \beta^2 - C^2\alpha^2 = 0$$

which is independent of η. It therefore represents the trace of the plane on the plane $\eta = 0$ and hence the required straight line in the $\xi\ \zeta$—plane.

The equations of transformation from the plane xy to the plane $\xi\ \zeta$ —plane are,

$$x = (\ \xi\ )\ (1 - \zeta\ )$$
$$y = ((\ \zeta\ \ (1 - \zeta\ )) - (\ \xi^2)\ (1 - \zeta\ )^2)^{1\ \ 2}$$

and the equations of transformation from the pseudosphere to this plane are,

$$v^2 + C^2e^{-2u}\ c = (\ \zeta\ )\ (1 - \zeta\ )$$
$$v = (\ \xi\ )\ (1 - \zeta\ )$$

Discussion of the Transformation.

The entire upper part of the xy—plane is represented inside the circle

$$\xi^2 + \zeta^2 \quad \zeta = 0$$

The circles $x^2 + y^2 - 2\beta x + \beta^2 - C^2\ \alpha^2 = 0$ become the straight lines

$$(\alpha^2 - \beta^2\alpha^2 + C^2)\ \zeta - 2\ \beta\alpha^2\ \xi + \beta^2\alpha^2 - C^2 = 0$$

The straight lines $y = k$ go into a sheaf of conics,

$(k^2 + 1)\zeta^2 - (2k^2 + 1)\zeta + \xi^2 + k^2 = 0$ through the point (0, 1). And since $-(k^2 + 1)$ is always negative the conics are all ellipses. The real part of the pseudosphere is therefore represented in the area included between the ellipses corresponding to the lines $y = C$ and $y = C/e$.

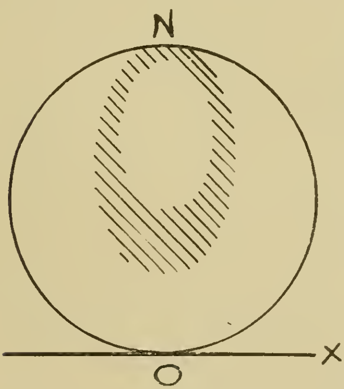

Fig. 2.

All the ellipses are tangent to the circle at the point (0, 1) and have their foci on the ζ—axis. The circles concentric at the origin become the lines $\zeta =$ constant, chords parallel to the ξ—axis. The system of circles with centers on o—x and passing through the point a,b goes over into the system of straight lines through the point

$$\xi = (a)\ \ (a^2 + b^2 + 1)$$
$$\zeta = (a^2 + b^2)\ \ (a^2 + b^2 + 1)$$

Two such systems properly related and having the point (a,b) on the same line $y = b$ go over into the two projectively related sheaves of lines whose corresponding rays intersect on the conic corresponding to $y = b$. In particular, in case the points (a,b) are on the x—axis the conic becomes the circle o—b and the corresponding rays are at right angles. Circles with the centers on the x—axis and of equal radii go over into the straight lines enveloping an ellipse. The line $x = 0$ goes into $\xi = 0$ the points being moved along the line. The origin is the fixed point of transformation.

Asymptotic Lines on S.

The asymptotic lines on the surface are defined by the equation

$$D\,du^2 + .2D'du.dv + D''dv^2 = 0 \qquad \text{(See Note 1)} \quad \ldots\ldots (12)$$

This becomes for the surface S,

$$C + (C^2 - e^{2u}/c)^{1/2} = e^{u}/c\,e(\mp v + \beta) \qquad \ldots\ldots (13)$$

and by (8) becomes in the x—y plane

$$y = -(y^2 - 1)^{1/2} + e(\mp x + \beta) \qquad \ldots\ldots (14)$$

Loxodromic Lines on S.

The differential equation of the loxodromic lines of a surface are given *by

$$((E)^{1/2}/(G)^{1/2})\ .\ (du/dv) = \tan \alpha \qquad \ldots\ldots (15)$$

Where α is the constant angle which the curves make with the curves $v =$ constant. For S (15) becomes,

$$(C du/u^2) = \pm \tan \alpha\,.dv.$$

Hence $\qquad \tan \alpha\,.uv + k_1 u + C = 0$

This by the relation $u = e^{u'/c}$ becomes,

$$\tan \alpha\,.e^{u/c} v + k_1 e^{u/c} + C = 0 \qquad \ldots\ldots (17)$$

which by (8) gives,

$$y = -\tan \alpha\,.x - k_1 \qquad \ldots\ldots (18)$$

This is a system of straight lines parallel to the line

$$y = -\tan \alpha\,.x$$

and so a system of lines making a constant angle with the lines $x =$ constant. And this is as it should be since the geodesic lines $v =$ constant go over into the lines $x =$ constant by the same transformation.

By selecting lines from different systems of loxodromic lines we may envelop any geodesic except the meridians. This may be seen by changing (17) to the form,

$$x \sin \alpha + y \cos \alpha + k_1 \cos \alpha = 0$$

Where if k_1 and $\cos \alpha$ change so that $k_1 \cos \alpha =$ constant we get a system of lines enveloping a circle with the centers at the origin. This corresponds to the loxodromic lines on the surface enveloping a geodesic.

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*Bianchi, p. 109.

CHAPTER II.

Lines on the Syntractrix of Revolution.

Taking the equation of the syntractrix in the form,

$$x = (d^2 - y^2)^{1/2} + C \cosh^{-1} (d/y) \qquad \ldots\ldots(19)$$

the surface S is given by,

$$\left.\begin{aligned} x &= u \cos v \\ y &= u \sin v \\ z &= -(d^2 - u^2)^{1/2} + C \cosh^{-1} (d/u) \end{aligned}\right\} \ldots\ldots(20)$$

or we may transform the equation of the tractrix by

$$\left.\begin{aligned} y &= (C/d)y_1 \\ x &= x_1 + ((d - C)/d)(d^2 - y_1^2)^{1/2} \end{aligned}\right\} \ldots\ldots(21)$$

Giving as the relation between the surfaces S and S_1,

$$u = (c/d)u_1$$

$$v = v_1$$

In this work C represents the length of the tangents to the tractrix and d the constant distance taken on these tangents to get the syntractrix. Hence d = constant.C

We get for the fundamental qualities:

$E_1 = (u^2 - Cd)^2/(u^2(d^2 - u^2)) + 1$, $F_1 = 0$, $G_1 = u^2$ and

$D_1 - (u^2(d^2 - 2Cd) + Cd^3)/(u(d^2 - u^2)^{3/2})$

$$D'_1 = 0\ ,\ D''_1 = (u(u^2 - Cd))/(d^2 - u^2)^{1/2} \qquad \ldots\ldots(22)$$

$K_1 = ((u^2 - od)(u^2(d^2 - 2Cd) + Cd^3))/((d^2 - u^2)(u^2(d^2 \ \ 2cd) + C^2d^2)$

(Above equation is number 23 and is the equation of the Gaussian curvature.)

When C = d , (23) becomes —1 or the curvature of the pseudosphere. When $C = d/2, K_1$ becomes $(2u^2 - d^2)/(d^2 - u^2)$

Since for the surface $d \geqq u$ the denominator is always positive and the numerator is positive or negative according as

$$2u^2 - d^2 \gtrless 0$$

That is, according as $u > (d/(2)^{1/2})$ and $u < -(d)/((2)^{1/2})$ or $-(d/((2)^{1/2}) < u < d/((2)^{1/2})$. For $u = \pm d/((2)^{1/2})$, $K_1 = 0$. This means that for the particular surface S_1 defined by d = 2C the Gaussian curvature is zero for the circles u = constant, given by taking the distance d on the tangent whose inclination to the z—axis is $\pi/4$ or $(3\pi)/4$. Tangents to the tractrix whose inclination to the z axis is something between $\pi/4$ and $(3\pi)/4$ give the curves u = coustant along which the surface have a negative curvature.

When C d 2 we have from (23) K_1 positive, negative or zero according as $(u^2 - cd) = 0$. But $C < d$ 2 gives Cd d^2 2, so that $u^2(Cd$ $d^2/2$ is the condition for the positive curvature. The curvature is zero or negative when $u^2 = cd$ d^2 2 $(u^2(d^2 - 2Cd) + Cd^3 = 0$ giving the imaginary values for u). This shows that the tangent line to the tractrix which gives the parabolic circle has a different slope than in the case where $d = 2C$, since in this case u d $(2)^{1}$ ^{2} 2, i. e. $\sin 0$ $(2)^{1}$ ^{2} 2.

When d $2C$ we might consider three cases viz., C d $2C$, $C = d$ or C d. It will only be noted here that when $C = d$ the surface S^1 is the same as the surface S and K^1 is therefore -1.

In any case $u^{2\prime} - Cd = 0$ gives the valves of u for which the tangent line to the curve C is parallel to the u—axis.

Geodesic Lines on S^1.

Using the method of the calculus of variations as outlined in Chapter I we get for the geodesic lines on the syntractrix of revolution,

$$Fv' = (u^2 dv)\ (E_1 du^2 + G_1 dv^2)^{1}\ {}^{2} = r$$

Here two cases may be considered according as

$$r = 0 \text{ or } r \pm 0$$

(1) When $r = 0$, then either $u = 0$ or $dv = 0$. But $u \pm 0$, hence $dv = 0$ and therefore $v =$ constant. That is the meridians are geodesic lines,

(2) When $r \pm 0$ we have

$$dv = ((r\ u^2)(u^2(d^2 - 2Cd) + C^2d^2)^{1}\ {}^{2}\ ((d^2 - u^2)(r^2 - u^2)^{1}\ {}^{2})dv$$

(The above equation is number 24.)

To reduce this expression on the right hand side to a convenient form substitute,

$$u^2(d^2 - 2Cd) + C^2d^2 = (C^2d^2t^2)\ (t^2 - 1) \qquad \ldots\ldots(25)$$

This may be written $u^2k + k_1 = (k^1t^2)\ (t^2 - 1)$ for convenience then,

$$dv = (-k^{3}\ {}^{2}r\,t^2\,dt)\ ((kr^2 + k_1)^{1}\ {}^{2}.\ (kd^2 + k_1)^{1}\ {}^{2}\ ((at^2 - 1)\ .$$
$$(bt^2 - 1))^{1}\ {}^{2}) \qquad \ldots\ldots(26)$$

Where $a. = (kd^2)\ (kd^2 + k_1)$ and $b = (kr^2)\ (kr^2 + k_1)$

When $r \pm 0$ we may consider two cases

When $r = d$ and $r \pm d$

When $r = d$ equation (26) becomes,

$$dv = (-k^{3}\ {}^{2}d_1dt)\ (kd^2 + k_1(at^2 - 1)) \qquad \ldots\ldots(27)$$

so that

$$v = (-k^{3/2}d/(a(kd^2+k_1)))(t+1/2(a)^{1/2}\log((a)^{1/2}t-1)/((a)^{1/2}t+1)) + \delta \qquad \ldots\ldots(28)$$

Eliminating "t" between (25) and (28) we have the the geodesic lines for $r = d$ given by

$$v = \frac{-(u^2k+k_1)^{1/2}}{u.d} - \frac{(kd^2+k_1)^{1/2}}{2d^2}\log\frac{d(u^2k+k_1)^{1/2}-u(kd^2+k_1)^{1/2}}{d(u^2k+k_1)^{1/2}+u(kd^2+k_1)^{1/2}} + \delta$$

(The above equation is equation 29.)

When $r \neq d$ (26) gives rise to an elliptic integral for the reduction of which we recall from the general theory of elliptic integrals. (See Note 1.)

$$R(x) = Ax^4 + 4Bx^3 + 6Cx^2 + 4B'x + A'$$
$$g_2 = AA' - 4BB' + 3C^2$$
$$g_3 = ACA' + 2BCB' - A'B^2\ AB'^2 - C^3$$

In this case we have,

$$R(t) = abt^4 - (a+b)t^2 + 1$$
$$g_2 = ab + (a+b)^2/12$$
$$g_3 = (-ab(a+b))/6 + (a+b)^3/216$$

We also have

$$R'(t) = 4abt^3 - 2(a+b)t$$
$$R''(t) = 12abt^2 - 2(a+b)$$

Substituting in (26)

$$t = \epsilon + (1/4\,R'(\epsilon))/(pu - 1/24\,R''(\epsilon)) \quad \text{(See Note 2)} \ldots\ldots(30)$$

Where ϵ is one of the roots of $R(t) = 0$. In this case take $\epsilon = 1/(a)^{1/2}$

then,
$$R'(1/(a)^{1/2}) = (2(b-a))/(a)^{1/2}$$
$$R''(1/(a)^{1/2}) = 2(b-a)$$

So that (30) may be written,

$$t = 1/(a)^{1/2} + ((b-a)/(2(a)^{1/2}))/(pu - p\underline{v})$$

when $p\underline{v} = (1/12)(5b - a)$ and therefore

$$abt^2 = b + (b(b-a))/(pu - p\underline{v}) + (1/4)((b(b-a)^2)/(pu - p\underline{v})^2$$

Recalling now that,

$$(p'\underline{v})^2 = 4p^3\underline{v} - g_2p\underline{v} - g_3 \qquad \ldots\ldots(31)$$
$$p''\underline{v} = 6p\underline{v} - 1/2\,g_2 \qquad \ldots\ldots(32)$$

and also,

$$(p'\underline{v})^2/(p\underline{u} - p\underline{v})^2 + (pu - p''\underline{v})/(p\underline{v}) =$$
$$p(\underline{u} + \underline{v}) + p(\underline{u} - \underline{v}) - 2p\underline{v} \qquad \ldots\ldots(33)$$

NOTE 1.—Klein, Modular, Functionen, Vol. I, p. 15.

NOTE 2.—Enneper, Elliptische Functionen, p. 30.

We get in the present case,

$$(p'v)^2 = ((b(b—a)^2) 4$$

$$p''v = b(b—a)$$

Equation (26) may be written,

$$v = ((-k^3\ ^2r)\ ((ab(kr^2+k_1)^{1/2})(kd^2+k_1\ \ ^2)\ .$$

$$\int(b = p(u+v) + p(u + v) + 2pv)bu + \delta$$

and so

$$v = K((1\ 6)(b—a)u + (\sigma'\ \sigma)(u + v) + (\sigma'\ \sigma)(u + v)) + \delta \qquad ......(34)$$

where $K = (-(k)^{1/2})/(d(ab)^{1/2})$

The geodesic lines on S are then given by means of t,

$$u^2k + k_1 = (k_1t^2)\ (t^2 - 1)$$

$$v = K\ \phi\ t) + \delta$$

where $\phi(t)$ is given in (34) and $u = p^{-1}((b—a)\ (2(a)^1\ ^2t - 2) + pv)$

$$v = p^{-1}((5\ 12)b - (a\ 12))$$

If (24) be put in the form

$$(du\ dv) = (u^2\ r)\ ((d^2 - u^2)(r^2 - u^2))^1\ ^2\ (u^2(d^2 - 2Cd) + C^2d^2)^1\ ^2$$

it is seen at once that the equation is satisfied by the values $u =$ constant. But from the geometric consideration it is evident that, in general, the circles $u =$ constant are not geodesic lines since the normals to a geodesic line must also be normal to the surface. And from figures V and VI it is seen at once that this is only true for the circle $u = d$, where $d\ \ C$, and for the trivial case $u = 0$ no matter what the value of d.

The geodesic lines on the surfaces S_1 may be studied if the surfaces are divided into classes according as $d = 2C$.

In the case $d = 2C$ the general integral (26) takes the form,

$$v = \int((d^2r)\ (2u))\ ((du)/((d^2 - u^2)(r^2 - u^2))$$

which when $u = 1/t$ may be written as

$$v = - (—d^2r)/2 \int(t^2dt)/((d^2t^2 - 1)(r^2t^2 - 1)$$

Here $R(t) = d^2r^2\ t^4 - (d^2 + r^2)t^2 + 1$. It is evident that this is exactly the same as the $R(t)$ of the general case if we replace d^2 by a and r^2 by b. Taking note of this we may write the geodesic lines in terms of t

$$u = 1\ t$$

$$v = (—1\ 2r)\ (1\ 6\ (r^2—d^2)u + (\sigma'\ \sigma\ (u+v) + (\sigma'\ \sigma)(u-v)) + \delta$$

where $u = p^{-1}((r^2—d^2)/(2dt—2) + pv)$ and $v + p^{-1}(5\ ^2— d^2)/(12)$. In this case the geodesics are real for all values of r.

In particular when $d = 2C$ and $r = d$ (29) becomes

$$v = \overline{+} (d/2u) \overline{+} (1/4) \log (d-u)/(d+u) + \delta$$

For the purpose of illustration let $d = 1$ then (35) becomes

$$v = \mp (d/2u) \mp (1/4) \log (1-u)/(1+u) - \delta$$

And since δ is an added constant we may without loss of generality let $\delta + 0$.

This particular geodesic line has been drawn in figure 3. It is to be noted that the line winds around the surface as it approaches smaller values, and then again winds around approaching the circle $u = 1$. The lines $r = d = 1$ are all similar to this one and may be obtained by giving different values to δ .

When $d > 2C$, $k = (d^2 - 2Cd)$ is positive and ab is positive and since $k_1 = C^2d^2$ is always positive and we have K always real so that the geodesic lines on the surface S_1 defined by $d > 2C$ are all real.

When $d < 2C$, $k = (d^2 - 2Cd)$ is negative and ab is positive or negative according as $r^2 \gtrless \left| k_1/k \right|$ or $\gtrless \left| C^2d^2)/(d^2 - 2Cd \right|$ So that on the surface S_1 defined by $d < 2C$, K will be real or imaginary according as $r^2 \gtrless (k_1/k)$. Hence the geodesic lines on such surfaces become imaginary lines when $r^2 > \mid k_1/k \mid$, that is when $r > \mid k_1/k \mid ^1/^2$ and $r < - \mid k_1/k \mid ^2/^2$.

Asymptotic Lines on S_1.

From the general equation of the asymptotic lines on a surface we get for the asymptotic lines on S_1,

$$(u^2(d^2 - 2Cd) + Cd^3)^1/^2/(u((Cd - u^2)(d^2 - u^2)^1/^2)) \, du = \underline{+} \, dv$$

(The above equation is number 37).

The substitution of $u^2(d^2 + 2Cd) - Cd^3 = 1/t^2$ reduces (37) to the form,

$$(-kdt) \; ((1-k_1t^2)((at^2 - 1)(bt^2 - 1))^{1\ 2} - \underline{+} \, dv.$$

Where $k = d^2 - 2Cd$, $k_1 = Cd^3$, $a = Cdk + k_1$, $b = d^2k + k_1$.

In the particular case when $d = 2C$ (37) becomes

$$((d^2) \; (u((d^2 - 2u^2)(d^2 - u^2))^{1\ 2})) \, . \, du = + \, dv$$

Which when $u = 1 \ t$ reduces to

$$(-d^2t \, . \, dt) \; ((d^2t^2 - 2)(d^2t^2 - 1))^{1\ 2} = \underline{+} \, dv \qquad \ldots\ldots(39)$$

Here $\quad R(t) = d^4t^4 - 3d^2t^2 + 2$

$R'(t) = 4d^4t^3 - 6d^2t$

$R''(t) = 12d^4t^2 - 6d^2$

$g_2 = (11/4)d^4$

$g_3 = (9/8)d^6$

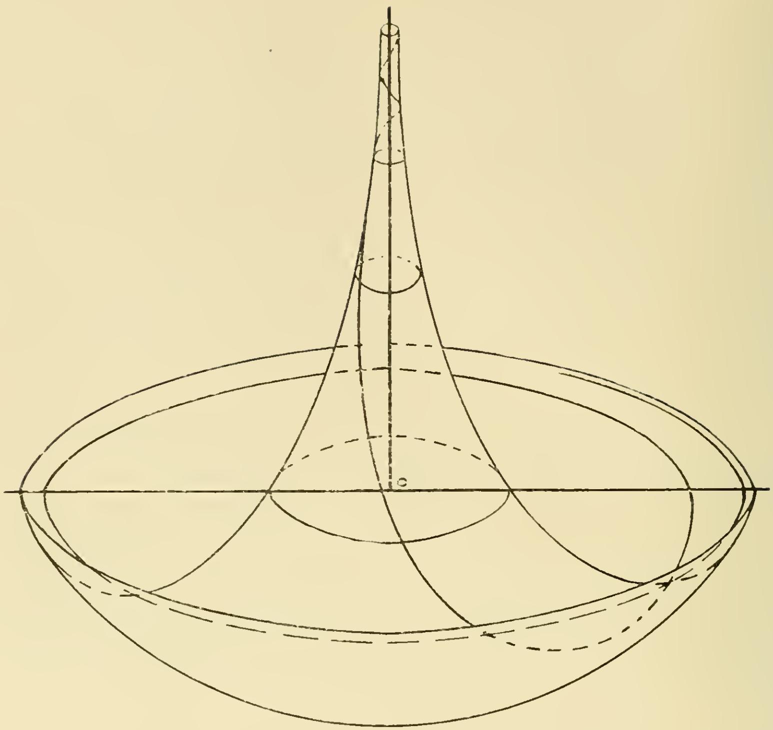

Fig. - 3

To reduce (39) substitute

$$t = (a + (1/4R'(a))\ (pu - (1/24)R''(a)) \qquad \ldots\ldots (40)$$

Where a is a root of R(t). In this case take $a = 1/d$. Then equation (40) may be written,

$$t = ((1/d) + ((-d/2)/(pu - pv)) \qquad \ldots\ldots (41)$$

where $pv = d^2/4$

Since $(p'v)^2 = 4p^3v - g_2 pv - g_3 - ((2)^{1/2})/2)d^3$ and $(-p'v)/(pu - pv) = (\sigma'/\sigma)(u + v) - (\sigma'/\sigma)(u - v) - 2(\sigma'/\sigma)(v)$ (Note 1) we have, remembering the relation $(dt/du) = (R(t))^{1/2} + v = (-1/((2)^{1/2}) \int ((2)^{1/2}d + \frac{\sigma'}{\sigma}$

$-(u + v)(\sigma'/\sigma)(u - v) - 2(\sigma'/\sigma)v)du + \delta' + v = (-(d - (2)^{1/2}(\sigma'/\sigma)(v)u)$

$- ((2)^{1/2})/2 \log (\sigma(u - v)/\sigma(u + v) + \delta'$

(The above is equation 42.)

Note: Schwarz. Formeln der elliptschen Functionen, p. 13.

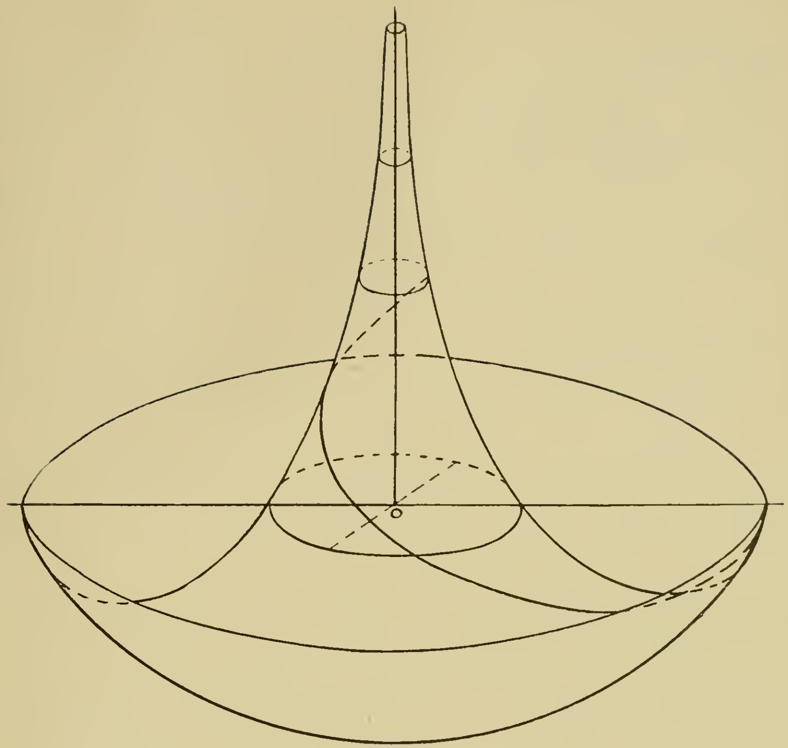

Fig. — 4

The asymptotic lines in this case are then given by the equations,

$$u = 1 \; t$$

$$v = \Psi(t) + \delta'$$

where $\Psi(t)$ is given in (42) & $u = p \;\; ^{1}((3d^2 - td^3) \; (4 - 4td))v = p^{-1}(d^2 \; 4)$

Loxodromic Lines on S^1.

The general equations for the loxodromic lines on a surface $((E)^{1/2} \; (G)^{1 \; 2})$, $du = \pm \tan \alpha \, dv$ becomes in the case of S_1 $((u^2(d^2 - 2Cd) + C^2d^2)^{1/2}) \; (u^2(d^2 - u^2)^{1 \; 2}) \;)du = \pm \tan \alpha \cdot dv$ which by the substitution $u^2(d^2 - 2Cd) + C^2d^2 = (C^2d^2t^2) \; (t^2 - 1)$ reduces to the form,

$$((2C - d)(t^2dt)) \; ((t^2 - 1)((d^2 - 2Cd)t^2 - (d - C)^2)^{1 \; 2} = + \tan \alpha \cdot dv.$$

(The above equation is number 43.) This may be put in the form,

$$((2C - d) \; (k_2)^{1 \; 2}) \cdot ((t^2Ct) \; ((k^2t^4) - (k^2 + 1)t^2 + 1)^{1 \; 2} = \pm \tan \alpha \cdot dv.$$

(The above is equation 44.)

Where $k_2 = (d-C)^2$ and $k^2 = (d^2 - 2Cd)/(d-C)^2$

Here,

$$R(t) = k^2t^4 - (k^2+1)t^2 + 1$$

$$R'(t) = 4k^2t^3 - 2(k^2+1)t$$

$$R''(t) = 12k^2t^2 - 2(k^2+1)$$

$$g_2 = (1/12)(1 + 14\,k^2 + k^4)$$

$$g_3 = ((1+k^2)/(216))(1 - 34\,k^2 + k^4)$$

(44) may be reduced by the substitution,

$$t = 1 + ((k^2-1)/2)(pu - p\underline{v}) \qquad \ldots\ldots(45)$$

Where $\quad p\underline{v} = (1/12)(5k^2-1)$

Then $k^2t^2 = k^2 + (k^2(k^2-1))(pu - p\underline{v}) + ((k^2/4)(k^2-1)^2)(pu - p\underline{v})^2$ and since $dt/d\underline{u} = (R(t))^{1/2}$ we get by using (31), (32) and (33) $\underline{+}\tan \propto \cdot v - (2C-d)/((k_2)^{1/2}(k^2))((1/6)(k^2+1)\underline{u} + (\delta'/\delta)(\underline{u}+\underline{v}) + (\delta'/\delta)(\underline{u}-v)) + \delta''$ $\qquad \ldots\ldots(46)$

We have then the loxodromic lines on the surface S_1 given in terms of t by the equations,

$$u^2(d^2-2Cd) + C^2d^2 = (C^2d^2t^2)/(t^2-1)$$

$$v = \phi(t) + \delta''$$

where $\phi(t)$ is given in (46) and $u = p^{-1}((2/(k^2-1)(t-1)) + p\underline{v})$ $\underline{v} = p^{-1}((5k^2-1)/(12))$

Since $k_2 = (d-c)^2$ is always positive it is to be noted that $\phi(t)$ is always real.

In particular when $d = 2C$ the equation, the general equation for the loxodromic lines reduces to,

$$((d^2/2)/(u^2(d^2-u^2)^{1/2}))\,du = \underline{+}\tan\propto \cdot dv \qquad \ldots\ldots(47)$$

and therefore

$$(-(d^2-u^2)^{1/2}/2u) = \underline{+}\tan\propto \cdot v + \delta'' \qquad \ldots\ldots(47a)$$

and these by the substitution $((d^2-u^2)^{1/2}/2u) = y$, $v = x$ are given in the x–y plane by the straight lines,

$$y = \underline{+}\tan\propto x + \delta'' \qquad \ldots\ (48)$$

But this is the system of lines into which the loxodromic lines of the pseudosphere may be transformed. Hence the loxodromic lines on S and S_1 (when $d = 2C$) may be represented by the same set of straight lines in the plane.

Suppose $d = 2C = 1$ and $\delta'' = 0$ and the $\tan\propto = 1$. Then 47_a becomes

$$(-(d^2-u^2)^{1/2})/(2u) = \underline{+}\,v.$$

This gives a line on the surface from the point $(u_1, v_1) = (1, 0)$ making an angle of 45° with the lines $v =$ constant. The line winds about the surface as shown in figure IV.

The surfaces S_1 might have been classified according as $d \gtreqless C$. The advantages of such a classification are not apparent in the analytical work and can only be seen from the geometry of the surface or the generating curve. In the work as presented the pseudosphere comes in as a special case of the surfaces S_1 when $d < 2C$, while if the classification had been made as above indicated the pseudosphere $d = C$ would be the dividing surface in the classification. On the whole I think the classification adopted is to be preferred. See figures V & VI for the different types of generating curves $d > C$, $d = C$ and $d < C$. The cut for $d < C$ is not given, but a general idea of the curve may be obtained by leaving off the loop in figure V.

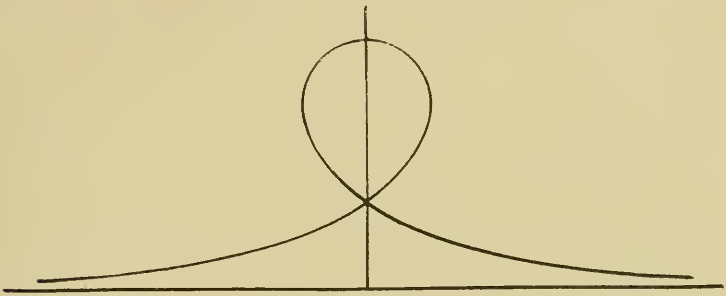

Fig.- 5

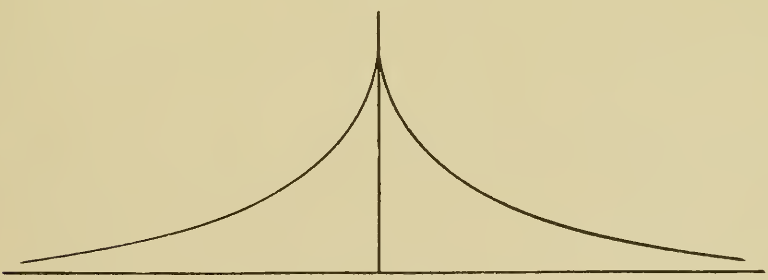

Fig.- 6