THE USE OF DETERMINANTS IN COLLEGE MATHEMATICS.

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In most American colleges it is customary to introduce the idea of the determinant in the college algebra course as another device whereby the solution of simultaneous linear equations in two or three unknowns may be obtained, and then in the remainder of the college course nothing more is heard of the determinant except possibly for a few lessons on the theory, that are soon forgotten, or for an occasional use in a course in the theory of equations or the analytic geometry of three dimensions.

That determinants should be introduced at all, considering the limited use that is made of them, is difficult to justify, unless the novelty of a new method of solving a most elementary part of algebra is sufficient, for, from a statistical study of answers made by Freshmen to questions on final examinations at Purdue, Mr. C. F. Barr,[1] of the Department of Mathematics, has deduced the result that the students are far more successful in obtaining correct solutions to simultaneous equations than in any other type of problem. He found that 94 per cent of the students under examination attempted the solution of simultaneous equations with an average grade of 7.6 out of a possible 10 on the question, whereas in such elementary operations as removing parentheses and carrying out other fundamental numerical processes, 95 per cent attempted the solution with an average grade of 5.5 out of a possible 10. The solution of the simple quadratic equation in one variable with numerical coefficients was attempted by only 85 per cent of the students, with a success of 6.3 out of a possible 10. Questions on factoring, simultaneous quadratics, linear fractional equations, logarithms, binomial expansion and complex numbers were attempted generally by a much smaller per cent of the students, with a corresponding lower degree of success. These results tend to confirm the belief that it is a pedagogical mistake to introduce determinants in college algebra merely to solve simultaneous linear equations, when students as a rule come prepared with other methods from their high school courses. Determinants at this point in college algebra certainly effect no time saving nor do they economize thought, which is one of the principal functions of mathematics, for none of their theory is attempted until much later, if at all, and the third order determinants are usually evaluated by a method that is of no use for any higher order determinants.

I am a strong believer in the use of determinants in college mathematics, but I do not believe they should be introduced merely for the sake of solving simultaneous equations. As a matter of economy in thought and time it would be better to leave the solution of simultaneous equations in two, three or more unknowns to the course in analytic

[1] Characteristic Errors in the Algebra of College Freshmen, Indiana Section of Math. Ass'n of America, May 9, 1925.

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geometry where a geometrical interpretation may be given to the algebraic solutions.

During the past two years, as an experiment, I have prefaced my course in plane analytic geometry with a few days of theory and drill in the fundamental properties and operations applicable to determinants of any order. After defining the rectangular co-ordinate system and plotting points as the intersections of straight lines, the points being considered as geometric solutions of simultaneous equations, the expression for the area of a triangle the co-ordinates of whose three vertices are given is developed as a determinant. It is then very easy for the student to see that when the area of the triangle is zero, the three vertex points are collinear, and hence the condition for collinearity of three points in a plane is that this determinant vanish. If one of these three points be a variable point, one immediately has the equation of the straight line determined by the other two points. Here one may now introduce the determinant method of solving two linear simultaneous equations and then find the condition for the concurrency of three straight lines, which is expressed by the vanishing of the determinant whose elements are the coefficients of the three equations. Thus one introduces the seeds of the principle of duality. It is also evident that the concepts of the parameter and undetermined coefficient may be introduced here through the use of some element of the determinant as an arbitrary or undetermined constant, to be determined by the conditions of the problem.

Passing from the study of the straight line to the study of the circle, the equation of the circle determined by three noncollinear points is expressed by a determinant of the fourth order equated to zero, and the expansion of the determinant is comparatively easy. Continuing into the conic sections the determinant again affords a convenient method of determining the nature of the conic, that is, the discriminant is most readily expressed as a determinant.

In the study of space geometry the extension of ideas by analogy is most forcibly brought out by the use of determinants. Thus one has the volume of a tetrahedron, the co-ordinates of whose vertices are known, expressed by means of a fourth order determinant. It is obvious that the condition that four points be coplanar is that this determinant vanish, and if one of these points be considered as variable, the equating of this determinant to zero gives the equation of the plane determined by the remaining three points. Likewise the simultaneous solution of three linear equations in three unknowns is easily handled as the point of intersection of three planes, and one may also study the nature of the intersection of three planes by means of determinants and thus prepare the way for the study of matrices and the rank of a matrix. The condition that four planes be concurrent is easily expressed by means of a fourth order determinant. The student now passes to the study of the sphere and quadric surfaces with a working knowledge of determinants that is certain to be of great aid in advanced space geometry and the theory of equations. A comparatively easily expanded fifth order determinant equated to zero gives the equation of the sphere determined by four non-coplanar points, and the discriminant for the general second degree equation in three variables is a determinant of the fourth order. Thus the beauty and generality of certain fundamental procsses are emphasized. If time permits one may develop determinants which when equated to zero give the equation of a plane through a given point and parallel to two given lines, the condition that two given lines in space intersect, and other interesting geometrical relations in space.

The argument is sometimes advanced that the student does not acquire as much working strength in analytic geometry by the use of determinants as by other methods which generally are much more laborious and much less elegant. This same argument can be advanced against the use of any formula or short cut. Of course the mere substitution of values in a formula implies no special knowledge of the subject, but if the student is held responsible for the derivation of the formula, he must have a working knowledge of the subject.

The results of this experiment, so far as I can observe any results, are interesting. At Purdue University only 12 weeks are devoted to the study of plane analytic geometry and an introduction to space geometry, and the remaining six weeks of the semester are given to the study of either college algebra or trigonometry, so that the grade earned by a student is a composite grade based on 18 weeks of work and not on analytic geometry alone. However, the students under this experiment took the same final examinations as all other students pursuing the analytic geometry course.

In the first group of 21 students under the experiment, 15 passed the course, three failed outright and three were conditioned. Of the 15 students who passed, 13 returned to school the following year and took the calculus course. Twelve of these passed the differential calculus and ten passed the integral calculus. The two who did not pass the integral calculus were both conditioned and one removed the condition by special examination the following fall. It is evident that the percentage of success following the analytic geometry is relatively high.

The experiment was next tried on a class of students most of whom had failed previously in either algebra or trigonometry, or had failed in analytic geometry and were repeating the course. They were *irregulars*, mostly second year students. Of this class of 22, 19 passed the course and 17 of these received the lowest passing grade and none received the highest passing grade. All except one of those who passed took the differential calculus and eight passed the course and one ultimately removed his condition by special examination. This looks like a decidedly negative result, but it is not surprising, for, from a statistical study of the records of students who constitute these *repeating sections*, Prof. C. T. Hazard, of the Mathematics Department at Purdue, has found that approximately only ten per cent of these students ever graduate from Purdue. The moral is obvious and the result of the experiment on this class is not unexpected.

A third section composed of 23 students was experimented upon during the second semester of last year. Seventeen passed the course and two were conditioned in the trigonometry portion of the semester's work. One of these conditions was removed by examination. These students, if now in school, are at present taking the course in calculus, and of course no record of their success is yet available.

However, since these students took the same final examinations as the students taking the textbook course, and since the percentage of failures has been no higher, it is evident that the use of determinants has not been detrimental. Furthermore, the one section upon which the data are complete shows an unusually high percentage of success in the calculus. Of course these samples are not sufficient to justify any very definite conclusions, but it is hoped that the results may be of sufficient interest to encourage others to make the experiment.

It is well known to the trained mathematician that determinants are almost indispensable in the mathematics beyond the elementary calculus. Jacobians, Hessians and Wronkians are fundamental in the study of linear transformations, invariants, differential equations and higher geometry. The use of determinants in the dynamics of deformable bodies in homogeneous and pure strains, and in the general theory of oscillations, demonstrate their importance even to the mathematical physicist. The curl of a vector and the cross or skew product of two vectors, represented as simple third order determinants, afford excellent illustrations of the use of determinants as a means of expressing these important concepts in a simple and easily remembered manner. Certainly the use of determinants all through college mathematics is valuable and worthwhile.

Determinants have been used freely in the college courses of the past, as such textbooks as C. Smith's *Conic Sections* and *Solid Geometry*, and Williamson's *Differential Calculus* attest, but during the past 10 or 15 years there has been a decided tendency to ignore the use of determinants in the textbooks on elementary analytic geometry, which is quite in line with the apparent lowering of standards of the courses in college algebra, analytic geometry and even the calculus in many American colleges. Until the standards are again raised to their former level, or a more selective system of admission to college mathetics courses is developed, this beautiful algebra within the algebra will likely remain unknown to the average American college student and our long-suffering superior student will be denied the use of this powerful mathematical tool. I am convinced that once the farseeing college teachers, who teach mathematics as a means and not as an end, are freed from the necessity of catering to mass teaching, the use of determinants will again become general in college mathematics, and our superior students will find their progress into higher mathematics much accelerated.