NOTE ON THE LINES OF FORCE IN A PLANE ELECTROSTATIC FIELD CONTAINING TWO CHARGES.

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It is well known that the equations of the lines of force and the equations of the equipotentials in an electrostatic field containing two charges are respectively

$$(1).\quad \frac{e_1(x+1)}{r_1}+\frac{e_2(x-1)}{r_2}=C \qquad (2).\quad \frac{e_1}{r_1}+\frac{e_2}{r_2}=k$$

Charges e_1 and e_2 are located at A (—1,0) and B (1, 0) and are distant r_1, and r_2 from the point P (x, y). C and k are arbitrary constants.

$$\text{Thus } r_1=\sqrt{(x+1)^2+y^2} \quad \text{and} \quad r_2=\sqrt{(x-1)^2+y^2}$$

The direction of the lines of force at any point (x, y) is

$$\frac{dy}{dx}=\frac{y(r_1^3+r_2^3)}{(x-1)r_1^3+(x+1)r_2^3}$$

At the point A, r_1 vanishes. At B, r_2 vanishes. Therefore the above derivative becomes indeterminate at both A and B. This might have been expected from the fact that equation (1) is symmetrical with respect to the x-axis and that all lines of force must go through A and B. This means that A and B are singular points for equation (1), but it does not mean that each line of force has singular points. In fact, for each value of C equation (1) represents two lines of force. The author has not found a means of separating (1) into its two branches. In fact, after (1) is cleared of radicals it becomes an equation of the fourth degree in y^2.

The object of this paper is to obtain formulas which will determine the direction of the lines of force at A and B. To do this we might expand in a Taylor's series about A for instance, and examine the second order terms. This method, while elegant theoretically, presents practical difficulties which are apparently unsurmountable. Therefore the following more elementary method is used.

Equation (1) is cleared of radicals, the origin moved to A and the group of terms of lowest degree set equal to zero. The result is

$$e_1^4x^4+e_2^4(x^2+y^2)^2+C^4(x^2+y^2)^2-2e_1^2e_2^2x^2(x^2+y^2)-2c^2e_1^2x^2(x^2+y^2)$$
$$-2C^2e_2^2(x^2+y^2)=0.$$

This is quadratic in x^2 and in y^2. Solving as a quadratic in y^2 we get

$$y=\pm\sqrt{\frac{e_1^2(e_2\pm C)^2-(e_2+C)^2\,(e_2-C)^2}{(e_2-C)^2\,(e_2+C)^2}}$$

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so that the slopes of the lines tangent to (1) at A are

$$(3).\qquad \frac{\pm\sqrt{e_1^2-(e_2-C)^2}}{e_2-C},\qquad \frac{\pm\sqrt{e_1^2-(e_2+C)^2}}{e_2+C}$$

Similarly the slopes of the lines tangent to (1) at B are

$$(4).\qquad \frac{\pm\sqrt{e_2^2-(e_1-C)^2}}{e_1-C},\qquad \frac{\pm\sqrt{e_2^2-(e_1-C)^2}}{e_1+C}$$

Evidently (1) is symmetrical with respect to the x-axis so that it will be sufficient to consider only the upper half of the curve. Thus the double signs may all be taken positive.

To illustrate the use of these formulas, consider a field in which $e_1 = 4$ and $e_2 = -1$. Suppose a line of force leaves B at right angles to A B. Formulas (3) give

$\dfrac{\sqrt{1-(4-C)^2}}{4-C} = \tan 90°$ and $\dfrac{\sqrt{1-(4+C)^2}}{4+C} = \tan 90°$. Whence $C = \pm 4$. Substitute in (4) and get $\dfrac{\sqrt{16-(-1\pm 4)^2}}{-1\pm 4}$ or $\dfrac{\sqrt{7}}{3}$ and $-\dfrac{3i}{5}$.

Thus the real branch of this line of force enters A at an angle of about 42°.

These formulas may also be used to determine what must be the relative strength of charges at A and B in order that the field shall contain a certain line of force. Suppose that it is desired to produce a field such that a line of force which leaves A at an angle of 45° shall enter B at the same angle. Formulas (3) and (4) give respectively

$$C = \pm e_2 \pm \frac{e_1}{\sqrt{2}},\qquad C = \pm e_1 \pm \frac{e_2}{\sqrt{2}}.$$

in which the double signs may be chosen independently of each other.

Then $\pm e_2 \pm \dfrac{e_1}{\sqrt{2}} = \pm e_1 \pm \dfrac{e_2}{\sqrt{2}}$ from which $\dfrac{e_1}{e_2} = \dfrac{\pm\sqrt{2}\pm 1}{\pm\sqrt{2}\pm 1} = \pm 1, \pm 3 \pm 2\sqrt{2}.$

On account of squaring, the distinction between tan 45° and tan 135° has been lost sight of, but the desired field can be selected by substituting the above results in (3) and (4) and by using physical considerations.