THE EFFECT OF VARIATIONS OF THE FORCE FUNCTION UPON ORBITS OF LEAST ACTION.

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A unique episode in the history of astronomy resulted from a hypothesis of Leverrier. To account for the motion of the perihelion of Mercury he supposed there was a small planet (Vulcan) rotating between Mercury and the sun. As late as 1905 the opinion was tentatively accepted by many astronomers that the irregularities of motion were due to the presence in that region of a ring of meteoric substance. In the case of the analogous erraticism of the moon's motion around the earth Hall tested the effect of assuming a gravitational force μ/r^{2+e}, $e \doteq 0$, instead of Sir Isaac Newton's function μ/r^2. E. W. Brown showed (1903) that this alteration led to errors. In 1905 Einstein published the first of his work on relativity and somewhat later an equation of the orbit of a single planet which was soon verified for the case of Mercury.

The following postulates, which are probably the simplest possible, suffice for the present paper and we are led to an orbital equation of a form which includes that of Einstein as a special instance, although the postulates do not involve us with questions of relativity.

(a) Space is euclidian. Thus the squared element of (plane) arc is $ds^2 = dx^2 + dy^2$.

(b) A particle moving in a plane under the action of a central force (and no other influences except an initial velocity) moves according to the principle of least action.

(c) The force of attraction between a planet and the sun is of the form $\mu/(r^2 + \alpha r + \beta)$, r being the distance and α, β infinitesimals.

I. **Least action in a plane.** If a particle Q is projected from a point A, in a plane M, with an initial velocity v_0 and is attracted toward a fixed point O in M by a force P its motion is such that the action integral $\int v ds$ is a minimum. If m is the mass, O the cartesian origin and Q the point (x, y) the equations of motion are,

$$(1) \qquad m\ddot{x} = -Px/r, \quad m\ddot{y} = -Py/r, \quad (r = OQ)$$

If we multiply these equations by $2\dot{x}$, $2\dot{y}$ respectively and add we obtain, after integrating once,

$$v^2 = -2\int F dr + b, \quad (F = P/m, \ v = \dot{s}).$$

Taking for initial conditions $v = v_0$ when (r, θ) is (r_0, θ_0) and indicating by $\int_0$ that the initial coordinates are to be substituted after the integration, we obtain,

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(2) $$b = v_0^2 + 2\int_0 F dr,$$

$$\int v ds = \int_{\theta_1}^{\theta_2} \sqrt{-2\int F dr + 2\int_0 F dr + v_0^2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

The orbit of the particle is an extremal[1] of this integral.

If we abbreviate as follows:

$$\int v ds = \int_{\theta_1}^{\theta_2} f(\theta, r, r_1) d\theta, \quad (r_1 = dr/d\theta),$$

Euler's equation for extremal curves is

(3) $$f_r - \frac{d}{d\theta} f_{r^1} = O,$$

and since f is explicitly free from θ the calculus of variations gives at once the first integral of (3),

$$\frac{r^2}{(r^2 + r^{1^2})^{\frac{1}{2}}} \left(-2\int F dr + b)\right)^{\frac{1}{2}} = C.$$

Substitution of $u = 1/r$, $p = du/d\theta$, rationalization and differentiation once with respect to θ give the known form,

(4) $$\frac{d^2u}{d\theta^2} + u = \frac{1}{C^2u^2} F\left(\frac{1}{u}\right).$$

This is the equation for extremals of the type which is simplest with reference to the theory of differential equations. It is immediately integrable in the form,

(5) $$\left(\frac{du}{d\theta}\right)^2 = 2\int \left[\frac{1}{C^2u^2} F\left(\frac{1}{u}\right) - u\right] du + d,$$

and completely integrable in an explicit form involving quadratures.

II. Variations from the Newtonian law of gravitational attraction.

We now assume,

(6) $$F = \mu/(r^2 + \alpha r + \beta),$$

the numbers α, β being arbitrary but infinitesimal.

From (4),

(7) $$\frac{d^2u}{d\theta^2} + u = \frac{\mu/C^2}{\beta u^2 + \alpha u + 1}.$$

We write $C = 1$ leaving μ arbitrary. The general solution of (7) with the right hand side replaced by a constant K, is

$$u = A \sin\theta + B \cos\theta + K,$$

A, B being constants of integration, hence any integral curve of (7) is infinitesimally consecutive to a conic.

If we divide numerator by denominator on the right hand side of (7) and neglect, as higher infinitesimals, expressions of orders >2 in α, β we have, after integrating once,

(8) $$\left(\frac{du}{d\theta}\right)^2 = 2\mu[ku^5 + lu^4 + mu^3 + nu^2 + u + d], \quad (=2\mu U),$$

[1]Bolza, *Lectures on the Calculus of Variations* (1904).

where,

$$k = \tfrac{1}{5}\beta^2,\ l = \tfrac{1}{2}\alpha\beta,\ m = \tfrac{1}{3}(\alpha^2 - \beta),\ n = -\tfrac{1}{2}(\alpha + \mu^{-1}).$$

There results,

$$\sqrt{2\mu}(\theta + \epsilon) = \int du/\sqrt{U}.$$

This is in the form of a hyperelliptic integral but since terms in k, l, m are infinitesimal it can be evaluated in finite form. In fact,

$$\int du/\sqrt{U} = \int V^{-\frac{1}{2}}du - \tfrac{1}{2}\int V^{-\frac{3}{2}}(ku^5 + lu^4 + mu^3)du + \frac{3m^2}{8}\int V^{-\frac{5}{2}}u^6 du,$$

where,

$$V = nu^2 + u + d.$$

The method of integration by parts may be applied to each integral. If, as ordinarily, we neglect all expressions of order >1 in α, β and assume $u = Kx$, K being a constant whose value is determined when the linear unit is chosen then (8) becomes [2],

$$\left(\frac{dx}{d\theta}\right)^2 = -\tfrac{2}{3}\beta K\mu x^3 - (\alpha\mu + 1)x^2 + \frac{2\mu}{K}x + \frac{2\mu d}{K^2}. \tag{9}$$

Choose K according to the units employed and

$$\mu = K\lambda,\ d = \tfrac{1}{2}(E^2 - 1)K\lambda,\ \alpha = 0,\ \beta = -3/K^2\lambda,$$

and (9) becomes,

$$\left(\frac{dx}{d\theta}\right)^2 = 2x^3 - x^2 + 2\lambda x - \lambda^2(1 - E^2). \tag{10}$$

Now (10) is identical with Einstein's equation for the orbit of a single planet about the sun, M being the mass of the sun, *a* and E being respectively the major semi-axis and the eccentricity, and,

$$x = M/r,\ \lambda = M/a(1 - E^2).$$

For the planet Mercury the values in terms of Eddington units are

$$M = 1.45,\ a = 5.8.10^7,\ E = 0.206,\ \lambda = 2.6.10^{-8}.$$

It has been verified previously that the erratic advance of the perihelion of Mercury is in conformity with the equation (10). The erraticism of the motion of Venus has been explained by (10), (Eddington, loc. cit.) and probably may be accounted for more accurately by (8).

We have therefore developed the verified theory of the motion of planets without the intervention of relativity or of non-euclidian hypotheses, assuming only the principle of least action and a law of attraction (6) which differs infinitesimally from that of Newton.

[2]Eddington, *The Mathematical Theory of Relativity* (1923), p. 86.