Radiation from an Electron Using the Formula, H=Curl Idl/r.

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The formula, H= Curl Idl/r, which first came to my attention in an article by J. H. Dellinger (Scientific Paper of the Bureau of Standards, No. 354) applies to all aerial circuits used in radio transmission. The thought came to me to see how it would apply to an electron while radiating energy. When applying this formula to a radio circuit we assume that the term Idl/r has the components IdX/r, IdY/r and IdZ/r parallel to the three axes. The curl equation can be written in several ways. I shall write it as follows:

$$H_x = d/dy\ (IdZ/r) - d/dz\ (IdY/r)$$
$$H_y = d/dz\ (IdX/r) - d/dx\ (IdZ/r)$$
$$H_z = d/dx\ (IdY/r) - d/dy\ (IdX/r)$$

Taking the simple case of a vertical aerial the current is all in the Y direction and all terms are zero except the last part of the H_x equation and the first part of the H_z equation. If the point, P, in question is placed in the XY plane the first of these two drops out since z is zero. Since our radio current is alternating current our expression for the current, I, becomes $I_o \sin w(t-t')$, where t′ is the time required for the disturbance to travel from the aerial to the point, P. The time, t′ is equal to r/c. Where r is the distance and c is the velocity of light.

Then H_z or $H = d/dx\ (I_o dY/r) \sin w(t-r/c)$. Since $r = (x^2+y^2+z^2)^{1/2}$ the quantity, x, is involved in r and in differentiating the equation there will be two terms. Differentiating we get $H = -(I_o dy/r^2) \sin w\ (t-r/c) \cos \Theta - (I_o dY/r\ wc) \cos w(t-r/c) \cos \Theta$.

The first term or the sine term is induction and varies inversely as the square of the distance and is time phase with the current. The second term or the cosine term varies inversely as the first power of the distance. The first term is due to the velocity of the electron and the second term is out of phase with current and may be said to be due to the acceleration of the electron.

It can be shown that these two components have the same numerical value at a distance of $\gamma/2\pi$ from the areial. This is about one-sixth of a wave length and at a distance of a wave length or more the field is all radiation since for practical purposes induction is zero in comparison.

We may assume we have an electron vibrating harmonically in the Y, or vertical direction at the origin of the axes. The charge, e=Idt. Multiply both sides by dl and divide both sides by dt and we have, e dl/dt=Idl. Or Idl=ev. Where v is the velocity of the electron. If the electron is vibrating we have, $v = v_o \sin wt$ at the origin and $v = v_o \sin w(t-r/c)$ at the point, P.

Differentiating this last expression with respect to t we get the acceleration, $a = v_0 w \cos w(t - r/c)$.

If one substitutes ev_0 for I_0dl and a for the above expression in the last half of radiation term of the above equation for the field, H we get, $H = (ea/rc) \cos \Theta$ as the expression for the magnetic component. Since in e. m. c.g.s. units the electric field component, $E = Hc$, we have, $E = (ea/r) \cos \Theta$. Since energy density is equal to $(1/4\pi)HE$, we have $(1/4\pi)\ (e^2a^2/cr^2) \cos^2 \Theta$ for the energy per cubic centimeter. It will be apparent that the energy radiated or the field is symmeterical about the Y axis.

The energy radiated in one second will be located between a sphere of radius, r and a concentric sphere of radius $(r+c)$. The value of this energy radiated per second can be obtained by integrating over this volume the energy density as expressed above times dV. Where $dV = r \cos \Theta\ d\phi\ rd\Theta\ dr$.

Making the substitutions and indicating the integrations between the proper limits we get,

$$\text{the energy, } W = (1/4\pi)\ (e^2\ a^2/c) \int_0^{2\pi} d\Theta \int_0^{\pi/2} \cos^3\Theta\ d\Theta \int_r^{r+c} dr.$$

Integrating with respect to ϕ between limits we get the expression multiplied by 2π. Integrating with respect to r we get the expression multiplied by c, when limits are used. Integrating with respect to Θ we have for the upper hemisphere,

$$W = (1/2)\ (e^2a^2) \int^{\pi/2} (1 - \sin^2) \cos \Theta\ d\Theta = (1/2)\ (e^2a^2) [\sin O - 1/3 \sin^3 O]_0^{\pi/2}$$

$= 1 - 1/3)\ (e^2\,a^2)$ for the half sphere. For the total energy we have $W = 2/3\ (e^2\ a^2)$ ergs per second as the rate of radiation. Here e, is expressed in e.m. c.g.s. units. If e is expressed in e.s. c.g.s. units we have $2/3\ (e^2\ a^2/c^2)$.

J. J. Thomson "Electricity and Magnetism," p. 61, gives values for E and H which if integrated as above gives this result. Richardson "Electron Theory" p. 258 gives $e^2\ a^2/6\ c^2$. Dull and Plimton "Elements of Electro Theory" p. 167 gives $2/3(e^2a^2/c^3)$.