The Cayleyan Cubic.

By C. A. Waldo and John A. Newlin.

The Use of the Bicycle Wheel in Illustrating the Principles of the Gyroscope.

By Chas. T. Knipp.

(Published in the Physical Review, Vol. XII, No. 1, January, 1901.

The Cyclic Quadrilateral.

By J. C. Gregg.

PROBLEM.

The opposite sides of a quadrilateral FGHI inscribed in a circle, when produced, meet in P and Q; prove that the square of PQ is equal to the sum of the squares of the tangents from P and Q to the circle.—No. 80, page 470, Phillips and Fisher's Geometry.

SOLUTION.

(See Fig. I.)

On PO and QO as diameters draw circles (centers S and T) and cutting circle O in C, D, E and K. QK and PD are tangent to O. Through the points Q, F and G draw a circle cutting PQ in A. Then $\angle PHG = \angle GFI = \angle QAG$ $\therefore$ $\angle PAG$ is the supplement of $\angle PHG$ and PAGH is cyclic, and

$PQ.PA = PF.PG = \overline{PD}^2$ and

$PQ.QA = QH.QG = \overline{QK}^2$ and adding these two equations

$\overline{PQ}^2 = \overline{PD}^2 + \overline{QK}^2$ — Q. E. D.

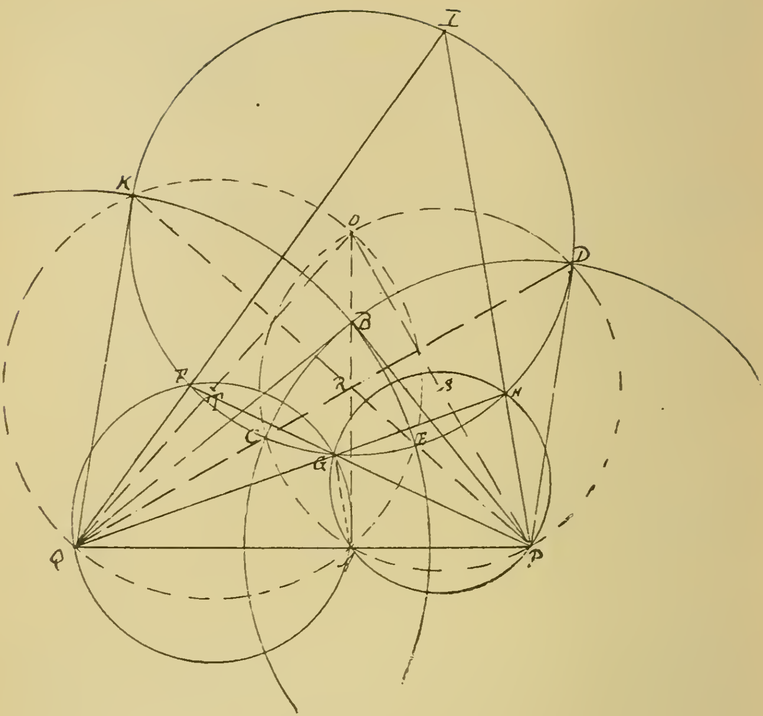

Fig. I.

DISCUSSION.

(1) With P and Q as centers, and PD and QK as radii draw two arcs meeting in B. Then PBQ is a right angle, and PB is tangent to arc EBK, and as the tangents PD and PB to circle O and arc EBK are equal, P must be on the common chord KE produced; and in the same way DCQ is a straight line.

(2) Since PK.PE=PF.PG=PQ.PA, the point A is in the circumference T, and OA is perpendicular to PQ, and A is also in the circumference S.

(3) PQ.PA=PF.PG=PI.PH. ∴ the points A, Q, I, H are concyclic, and in the same way A, P, I, F are also concyclic.

(4) PK and QD are respectively perpendicular to QO and PO, and R is the orthocenter of the triangle POQ, and AO passes through R.

(5) The three arcs DEC, EBK and DBC cut orthogonally, two and two, and the common chord of any two of them passes through the center of the third.

(6) (See Fig. II.)

$\angle GPA + \angle GQA = \angle HIF$ (supplement of $\angle PGQ$).

$\angle GPH = \angle GAH$.

$\angle GQF = \angle GAF$ and adding these three equations

$\angle QPI + \angle PQI = \angle HAF + \angle HIF$, or

$180° - \angle HIF = \angle HAF + \angle HIF$.

$180° - 2\angle HIF = \angle HAF$. But $\angle HOF = 2\angle HIF$.

$\therefore 180° - \angle HOF = \angle HAF$ and H, A, F, O, are concyclic.

(7) We have now shown the following points to be concyclic:

A, G, F, Q,—center M.
A, G, H, P,—center N.
A, O, K, Q,—center T.
A, P, D, O,—center S.
A, P, I, F,—center S′.
A, Q, I, H,—center T′.
A, H, O, F,—center O′.

And we will show that X is the center of a circle through A, G, O, I.

(8) CD, OA and HF are the three common chords of circles O, S and O′, and must meet in a point. Hence HF, the diagonal of FGHI, passes through R.

(9) Since APIF is cyclic $\angle QAF = \angle QIP$; and for the same reason $\angle PAH = \angle QIP$. $\therefore \angle QAF = \angle PAH$ and $\angle OAF = \angle OAH$.

(10) Since the circles S′, O′ and M pass through the points A and F, their centers S′, O′ and M are in the same line perpendicular to AF. For a similar reason N, O′, T′ are in the same line perpendicular to AH, and S′, S, N and T′, T, M are respectively in the same lines perpendicular to PQ or TS. Also T′S′, TS and MN respectively bisect AI, AO, and AG at right angles. Now the angles SO′S′ and SO′N have their sides respectively perpendicular to the sides of the equal angles OAF and OAH. $\therefore \angle SO'S' = \angle SO'N$ and $SN = SS'$, and in the same way $TM = TT'$. Hence the lines T′S′ and MN will meet TS at the same point X, and $XA = XG = XO = XI$ and X is the center of the circle through A, G, O, I.

(11) Now HF, OA, and GI are the three common chords of the circles O, O′ and X and must meet in a point. Hence GI the other

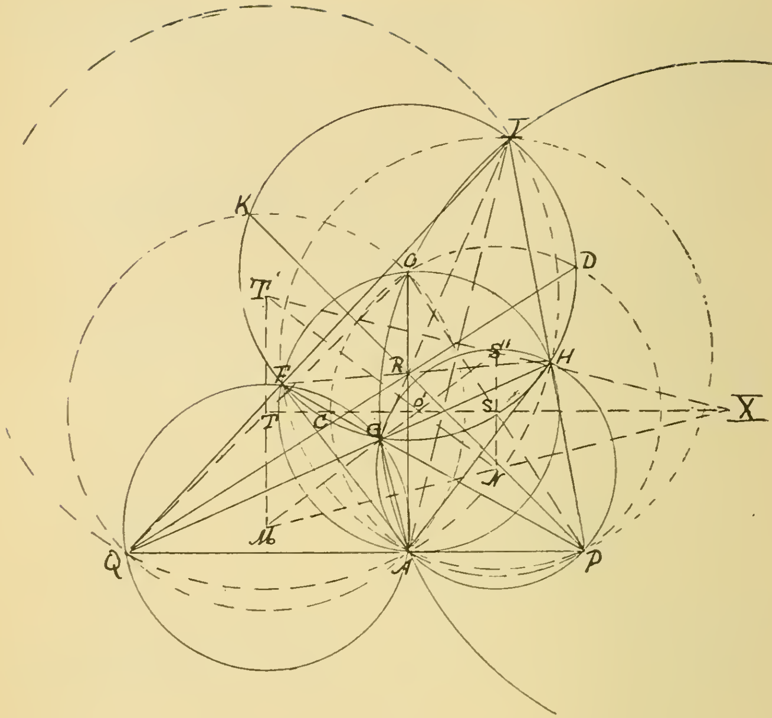

Fig II.

diagonal of FGHI also passes through R, and we have established the following

Theorem.—The diagonals of an inscribed quadrilateral meet in the orthocenter of the triangle whose vertices are the center of the circle, and the points where the opposite sides meet.

(12) (See Fig. 1.) Since QK, QE, PC and PD are tangents to circle O, the following theorem holds: If the diagonals of an inscribed quadrilateral meet in R, and its opposite sides meet in P and Q, and PR and QR be drawn cutting the circle in E, K, C and D, then PD, PC, QK and QE are tangent to the circle.

(13) The diagonals of *any* quadrilateral inscribed in circle O, and whose opposite sides meet in P and Q, will pass through R.

(14) If any point I, in circle O be joined to P and Q and cutting the circle in F and H, PF and QH will meet on the circumference as at G.