Note Relative to Peirce's "Linear Associative Algebra." James Byrnie Shaw, D. Sc.

I have no doubt many readers of Benjamin Peirce's classic work have found some difficulty in its perusal from the lack of examples of the algebras developed. That such a completion of the work was intended is shown by ¶ 2, p. 4, and the last three lines of page 119. The following method of exemplifying the subject may be of use or help. It is in a succinct form thus: Every unit in an algebra of this book is an operator of a matrical kind upon a ground of what we may call vectors. The whole work is thus a *treatise on groups of such operators.* This explains its abstruseness. Now for all cases in which the ground consists of two or three or four vectors, the units can be represented by the linear vector operators of quaternions, or linear quaternion operators. The relative forms given by Mr. C. S. Peirce may be immediately translated into such quaternion forms. Thus we may write, (α, β, γ, being vectors such that $S.\,\alpha\beta\gamma = 1$, and l_1, l_2, l_3, l_4, being quaternions such that $S.\,l_1\,A.\,l_2 l_3 l_4 = 1$ *).

Algebra $\mathbf{a}_1$, $i = \alpha S.\,\beta\gamma(\ )$.

" $\mathbf{b}_1$, $i = \alpha S.\,\gamma\alpha(\ )$.

" $\mathbf{a}_2$, $i = \alpha S.\,\beta\gamma(\ ) + \beta S.\,\gamma\alpha(\ )$; $j = \alpha S.\,\gamma\alpha(\ )$.

" $\mathbf{b}_2$, $i = \alpha S.\,\beta\gamma(\ )$; $j = \alpha S.\,\gamma\alpha(\ )$.

" $\mathbf{c}_2$, $i = \alpha S.\,\gamma\alpha(\ ) + \beta S.\,\alpha\beta(\ )$; $j = \alpha S.\,\beta\gamma(\ )$.

" $\mathbf{d}_2$, $i = l_1 S.(\ )\,A.\,l_3 l_4 l_1$; $j = l_3 S.(\ )\,A.\,l_1 l_2 l_3$.

" $\mathbf{a}_3$, $i = \alpha S.\,\beta\gamma(\ ) + \beta S.\,\gamma\alpha(\ ) + \gamma S.\,\alpha\beta(\ )$; $j = \alpha S.\,\gamma\alpha(\ ) + \beta S.\,\alpha\beta(\ )$;
$k = \alpha S.\,\alpha\beta(\ )$.

" $\mathbf{a}'_3$, $i = \alpha S.\,\beta\gamma(\ ) + \beta S.\,\gamma\alpha(\ )$; $j = \alpha S.\,\gamma\alpha(\ )$; $k = \alpha S.\,\alpha\beta(\ )$.

" $\mathbf{a}''_3$, $i = -l_1 S.(\ )\,A.\,l_3 l_4 l_1 - l_4 S.(\ )\,A.\,l_1 l_2 l_3$; $j = -l_1\ S.(\ )\,A.\,l_3$
$l_4 l_1$; $k = -l_3 S.(\ )\,A.\,l_1 l_2 l_3$.

" $\mathbf{b}_3$, $i = -l_1, S.(\ )\,A.\,l_3 l_4 l_1 + l_2 S.(\ )\,A.\,l_4 l_1 l_2 - l_3 S.(\ )\,A.\,l_1 l_2 l_3$;
$j = l_1 S.(\ )\,A.\,l_4 l_1 l_2 - l_2 S.(\ )\,A.\,l_1 l_2 l_3$; $k = -l_1 S.(\ )\,A.\,l_1 l_2 l_3$.

" $\mathbf{b}'_3$, $i = -l_1 S.(\ )\,A.\,l_3 l_4 l_1 + l_2 S.(\ )\,A.\,l_4 l_1 l_2$; $j = l_1 S.(\ )\,A.\,l_4 l_1 l_2$;
$k = -b_3 l_1\ S.(\ )\,A.\,l_1 l_2 l_3 + l_4 S.(\ )\,A.\,l_4 l_1 l_2$.

" $\mathbf{c}_3$, $i = -l_1 S.(\ )\,A.\,l_3 l_4 l_1 + l_2 S.(\ )\,A.\,l_4 l_1 l_2$; $j = l_1 S.(\ )\,A.\,l_4 l_1 l_2$;
$k = -\alpha\, l_1 S.(\ )\,A.\,l_3 l_4 l_1 - l_1 S.(\ )\,A.\,l_1 l_2 l_3 + l_4 S.(\ )\,A.\,l_4 l_1 l_2$.

" $\mathbf{d}_3$, $i = \beta S.\,\alpha\beta(\ )$; $j = \alpha S.\,\alpha\beta(\ )$; $k = \alpha S.\,\gamma\alpha(\ )$.

" $\mathbf{e}_3$, $i = -l_1 S.(\ )\,A.\,l_1 l_2 l_3$; $j = -l_1 S.(\ )\,A.\,l\,l_4 l_1 + l_3 S.(\ )\,A.\,l_1 l_2 l_3$;
$k = l_1 S.(\ )\,A.\,l_4 l_1 l_2 - l_2 S.(\ )\,A.\,l_1 l_2 l_3$.

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*) $A.\,l_2 l_3 l_4 = S.\,Vl_2 Vl_3 Vl_4 - Vl_2.\,V.\,Vl_3 Vl_4 - Vl_3.\,V.\,Vl_4 Vl_2 - Vl_4\ V.\,Vl_2 Vl_3$.
$S.\,l_1\,A.\,l_2 l_3 l_4 = -S.\,l_2\ A.\,l_3 l_4 l_1 = S.\,l_3\ A.\,l_1 l_1 l_2 = -S\,l_4\ A.\,l_1 l_2 l_3$.

Algebra g_4, $i = a\,S.\,\beta\gamma(\ )$; $j = a\,S.\,\gamma a(\ )$; $k = \beta\,S.\,\beta\gamma(\ )$; $l = \beta\,S.\,\gamma a(\ )$.

" bp_5, $i = l_2\,S.(\ )\,A.\,l_4l_1l_2 - l_3\,S.(\ )\,A.\,l_1l_2l_3$; $j = -l_2\,S.(\ )\,A.\,l_1l_2l_3$; $k = -l_1\,S.(\ )A.\,l_3l_4l_1 - l_3\,S.(\ )A.\,l_1l_2l_3$; $l = l_1\,S.(\ )A.\,l_4l_1l_2$; $m = -l_1\,S.(\ )\,A.\,l_1l_2l_3$.

" bk_6, $i = l_1\,S.(\ )\,A.\,l_2l_3l_4 - l_2\,S.(\ )\,A.\,l_3l_4l_1 + l_3\,S.(\ )\,A.\,l_4l_1l_2$; $j = -l_1\,S.(\ )\,A.\,l_3l_4l_1 + l_2\,S.(\ )\,A.\,l_4l_1l_2$; $k = l_1\,S.(\ )\,A.\,l_4l_1l_2$; $l = -l_3\,S.(\ )\,A.\,l_1l_2l_3$; $m = -l_2\,S.(\ )A.\,l_1l_2l_3$; $n = -l_1\,S.(\ )\,A.\,l_1l_2l_3$.

bm_6, $i = a\,S.\,\beta\gamma(\ )$; $j = a\,S.\,\gamma a(\ )$; $k = a\,S.\,a\beta(\ )$; $l = \beta\,S.\,\beta\gamma(\ )$; $m = \beta\,S.\,\gamma a(\ )$; $n = \beta\,S.\,a\beta(\ )$.

These examples can be used to illustrate the general theorems. For example: "*Every group of linear vector operators contains at least one idempotent or one nilpotent expresssion.*"

The group $b\,m_6$ contains the idempotents

$$a\,S.\,\beta\gamma(\ ),\quad \beta\,S.\,\gamma a(\ ),\quad a\,S.\,\beta\gamma(\ ) + \beta\,S.\,\gamma a(\ ).$$

The group $b\,p_5$ contains only nilpotents.

"*When an algebra contains an idempotent expression it may be assumed as the basis and the remaining expressions are then divisible into four classes.*"

In $b\,m_6$ if we assume $a\,S.\,\beta\gamma(\ )$ as the idempotent then the units are, with reference to the basis,

idemfaciend, idemfacient, $a\,S.\,\beta\gamma(\ )$;
nilfaciend, idemfacient, $\beta\,S.\,\beta\gamma(\ )$;
idemfaciend, nilfacient, $a\,S.\,\gamma a(\ )$, and $a\,S.\,a\beta(\ )$;
nilfaciend, nilfacient, $\beta\,S.\,\gamma a(\ )$, and $\beta\,S.\,a\beta(\ )$.

"*The fourth class are subject to independent investigation.*"

"*If the first class comprises any units except the basis, there is, besides the basis, another idempotent expression or a nilpotent expression, and we may free the class from this, when idempotent, by writing for the basis the difference between the two; in this case expressions may pass from idemfaciend to nilfaciend or from idemfacient to nilfacient. but not the reverse.*" Thus, if we had taken for our basis in $b\,m_6$ $a\,S.\,\beta\gamma(\ ) + \beta\,S.\,\gamma a(\ )$ there would have been only two classes,

1: $a\,S.\,\beta\gamma(\ ) + \beta\,S.\,\gamma a(\ )$; $\beta\,S.\,\gamma a(\ )$; $a\,S.\,\gamma a(\ )$; $\beta\,S.\,\beta\gamma(\ )$;
2: $a\,S.\,a\beta(\ )$; $\beta\,S.\,a\beta(\ )$.

The second idempotent basis is easily seen to be $\beta\,S.\,\gamma a(\ )$, and the difference is $a\,S.\,\beta\gamma(\ )$, as before. And making this change of basis, $\beta\,S.\,\gamma a(\ )$ and $\beta\,S.\,a\beta(\ )$ become fourth class, $\beta\,S.\,\beta\gamma(\ )$ becomes second class, $a\,S.\,\gamma a(\ )$ becomes third class.

"*When there is no idempotent basis, all expressions are nilpotent, and all powers of each expression that do not vanish are independent. We may take any expression as the* *basis, but it is well to select one which has the most powers that do not vanish."* Thus in b_p, we take l_2 S. ( ) A. $l_1 l_1 l_2 - l_3$ S. ( ) A. $l_1 l_2 l_3$, whose square is l_2 S. ( ) A. $l_1 l_2 l_1$, the cube vanishing. This algebra is then of second order. If A, B are any two expressions of it,

$$A^2 B + A B^2 + A B A + B A B = 0.$$

These examples are sufficient to show the use of these forms in interpreting the subject. It remains only to show how they may be applied in a few cases. There are of course for every one of them two fields of application at once suggested by this method of writing them, viz.: linear transformations and homogeneous strains. E. g., the nilpotent algebra d_3. The general expression of this algebra is

$$\phi = x \beta \mathrm{S}.\, \alpha \beta (\ ) + \alpha \mathrm{S}. (y \mathrm{V} \gamma \alpha + z \mathrm{V} \alpha \beta) (\ ).$$

This transforms $\rho = x_1 \alpha + y_1 \beta + z_1 \gamma$ into

$$\phi \rho = x z_1 \beta + \alpha (y y_1 + z z_1)$$
$$= y y_1 \alpha + z_1 (z \alpha + x \beta).$$

This may represent any point of the plane (α, β). Since the value of x_1 does not enter $\phi \rho$, every straight line parallel to α is made to correspond to a configuration of the (α, β) plane. Those lines parallel to α which cut the (β, γ) plane in a line parallel to β, correspond to a series of configurations of the (α, β) plane produced by slipping it along the direction α. The movement of a line which is parallel to α along a line parallel to the line γ, produces a series of expansions of the (α, β) plane from a point $y y_1 \alpha$ as center. If both y_1 and z_1 vary, subject to a law, we have the configuration of the (α, β) plane

$$\phi \rho = y y_1 \alpha + f(y_1)(z \alpha + x \beta).$$

Again, consider the algebra a_3. The general expression here, is

$$\phi = x (\alpha \mathrm{S}.\, \beta \gamma (\ ) + \beta \mathrm{S}.\, \gamma \alpha (\ ) + \gamma \mathrm{S}.\, \alpha \beta (\ )) + y (\alpha \mathrm{S}.\, \gamma \alpha (\ ) + \beta \mathrm{S}.\, \alpha \beta (\ ))$$
$$+ z \alpha \mathrm{S}.\, \alpha \beta (\ ),$$
$$= \alpha \mathrm{S}. (x \mathrm{V} \beta \gamma + y \mathrm{V} \gamma \alpha + z \mathrm{V} \alpha \beta) (\ ) + \beta \mathrm{S}. (x \mathrm{V} \gamma \alpha + y \mathrm{V} \alpha \beta) (\ )$$
$$+ \gamma \mathrm{S}.\, x \mathrm{V} \alpha \beta (\ )$$

ρ becomes $\phi \rho = \alpha (x x_1 + y y_1 + z z_1) + \beta (x y_1 + y z_1) + x z_1 \gamma$.

This strain operator will convert ρ into any other vector σ, for if

$$\sigma = \xi \alpha + \eta \beta + \zeta \gamma$$

we have at once from

$$\phi \rho = \sigma,$$
$$x x_1 + y y_1 + z z_1 = \xi,$$
$$x y_1 + y z_1 = \eta,$$
$$x z_1 = \zeta.$$

Whence

$$x = \zeta / z_1,$$

$$y = \frac{\eta z_1 - \zeta y_1}{z_1^2},$$

$$z = \frac{\xi z_1^2 - \zeta (x_1 z_1 - y_1^2) - \eta y_1 z_1}{z_1^3}.$$

The exceptional cases are where $z_1 = 0$. That is, ϕ can be so chosen as to convert any vector into any other except those lying in the plane of (α, β), which is converted into itself, the line $x_1 \alpha$ being converted into itself. The cubic of ϕ is $(\phi - x)^3 = 0$. We may write $\phi \rho = x \rho + (y y_1 + z z_1) \alpha + y z_1 \beta$.

Hence the effect of any ϕ is to move the terminal point of ρ along its line in either direction, and then slide this extremity along a plane parallel to (α, β). Thus the infinite number of strains, which belong to this infinite group of strains, and that have the same x, represent a group of shears. Space nor time permit a fuller treatment of this interesting line of application of this algebra. The application of the other algebras might similarly be deduced.

I may say in closing that the natural classification of these algebras referred to by Professor Benjamin Peirce, who regarded his own classification as Linnean, is pointed to by these representations of the algebras.

Illinois College, Dec. 23, 1895.

Variation of a Standard Thermometer. By Chas. T. Knipp.

During the term just past I made a number of observations on a standard thermometer. The problem that presented itself was to observe the variations in a standard thermometer under given conditions, and the minimum limit of conditions that would produce the same.

Having a delicate cathetometer at hand, that reads directly to $\frac{1}{50}$ and accurately to $\frac{1}{100}$ of a mm., no hesitancy was felt in making the observations, feeling assured that the slightest variations in the reading of the thermometer could be detected.

The thermometer that was in question was one of Queen & Co's standardized thermometers of the centigrade scale, graduated in tenths over a range of 100 degrees. The bulb is cylindrical in form, thus having a maximum, or tending towards a maximum surface and consequently increased sensitiveness.

The thermometer was tested and standardized by the above named company on the 10th of October. After standardizing it was put in a brass case lined with