rods, and are provided with pinions, cranks and binding screws to make accurate adjustment. The stand is provided with castors so adjusted that it may be thrown on or off its legs with the foot. All the handles are nickel-plated and the whole apparatus enameled black.

With this apparatus it is possible to work in the vertical or horizontal position or at any inclination. The adjustment is easily and quickly made by loosening the binding nut between the friction plates and turning the bed plate to the desired position. The bed plate can be rotated on the horizontal axis to get the advantage of room and direction of light without moving the stand upon its legs. When the bed plate is turned to the horizontal the top of the bed plate is 33 inches from the floor; too low to work with comfort. By raising the elevating post the bed plate may be carried up to the height of five feet. This adjustment makes it possible to always have the work at a comfortable height, either in the sitting or standing position, and regardless of the stature of the operator.

The apparatus has been used for some months in the Veterinary Laboratory of Purdue University. A Zeiss microscope and a long-focus premo camera are mounted upon the carriages (any other microscope and camera can be mounted as easily), and photographs have been taken of parasites, histological sections and bacteria. It has been used in all positions, with fresh and permanent mounts, and the results are entirely satisfactory.

The stand was built by C. W. Meggenhoffen, of Indianapolis.

An Infinite System of Forms, Satisfying the Requirements of Hilbert's Law. By J. A. Miller.

Let Z represent the totality of homogeneous integral forms of four variables (excluding those which vanish identically) which are unchanged by the group of linear substitutions generated by the operators.

$$
\begin{array}{llll}
S: & Z_1' = Z_2 & \text{and } T: & Z_1' = Z_4 \\
 & Z_2' = Z_3 & & Z_2' = -\varepsilon^3 Z_3 \\
 & Z_3' = \varepsilon^6 Z_1 & & Z_3' = \varepsilon^6 Z_2 \\
 & Z_4' = Z_4 & & Z_4' = -Z_1
\end{array}
$$

Where $\varepsilon = e^{\frac{2\pi i}{9}}$

Such a system exists. For inspection shows the elementary symmetric functions of Zi^6 $(i=1--4)$ are unchanged by the group, and hence $\overline{Z}$ consists of an infinite number of forms. The sequel will show, however, that these are not *all* the invariant forms of the group and hence we ask for an expression of the entire system.

1. *To determine analytic expressions for the system* $\overline{Z}$.

Definitions—An i-lettered form is one whose terms each contain i letters. By an invariant we shall mean a form invariant under the group.

An invariant may be expressed as a sum of

one-lettered forms,
two-lettered forms,
three-lettered forms, and
four-lettered forms.

Moreover, since the substitutions are linear, the sum of all i-lettered terms of an invariant is itself an invariant where i is any *one* of 1, 2, 3, or 4.

a. Four-lettered forms.

We shall consider first four-lettered invariants.

Let F be a four-lettered invariant. Where

$$F = Z_1^{\alpha} Z_2^{\beta} Z_3^{\gamma} Z_4^{\delta} + Z_1^{\alpha'} Z_2^{\beta'} Z_3^{\gamma'} Z_4^{\delta'} + Z_1^{\alpha''} Z_2^{\beta''} Z_3^{\gamma''} Z_4^{\delta''} + \ldots\ldots\ldots\ldots$$
$$\alpha+\beta+\gamma+\delta=\alpha'+\beta'+\gamma'+\delta'=\ldots\ldots\ldots\ldots$$

and where any exponent is a positive integer.

Suppose α is the least exponent, then

$F=(Z_1\, Z_2\, Z_3\, Z_4)^{\alpha}$ [a possible four-lettered form + a sum of i-lettered forms.]

Where $i=1---3$,

or $F=(Z_1\, Z_2\, Z_3\, Z_4)^{\alpha}\,[\phi_1+\phi_2]$ (say.)

Since $(Z_1\, Z_2\, Z_3\, Z_4)^{\alpha}$ is an invariant, so also is $\phi_1+\phi_2$, and therefore, by remark above, so also is ϕ_1. Hence ϕ_1 is susceptible of treatment similar to that applied to F, until finally we should have any four-lettered invariant expressed as the sum of invariant forms, of which a type is

$(Z_1\, Z_2\, Z_3\, Z_4)^{n}$ [sum of i-lettered forms] $i=1--3$.

Hence we need seek no expression for an i-lettered form where $i=3$.

b. Three-lettered invariant forms.

Let a term of this invariant be

$$Z_1^{\alpha}\, Z_2^{\beta}\, Z_3^{\gamma}$$

Apply S to this term and the terms resulting until no new terms are obtained, then forming the simplest symmetric function of these terms we reach

$$F_1 = [Z_1^\alpha Z_2^\beta Z_3^\gamma + \varepsilon^{3(\alpha+\beta)} Z_2^\alpha Z_3^\beta Z_1^\gamma + \varepsilon^{6(\beta+\gamma)} Z_3^\alpha Z_1^\beta Z_2^\gamma] \; [1 + \varepsilon^{3(\alpha+\beta+\gamma)} + \varepsilon^{6(\alpha+\beta+\gamma)}]$$

This form is invariant under S, but vanishes identically unless $\alpha+\beta+\gamma \equiv o$ (mod 3). Hence we conclude $\alpha+\beta+\gamma \equiv 0$ (mod. 3)...................... (1)

Apply T to F_1. Immediately there appears, among others, the terms $A\, Z_4^\alpha Z_3^\beta Z_2^\gamma$, and $B\, Z_3^\alpha Z_2^\beta Z_4^\gamma$, A and B being independent of Z; hence terms of this type are in our invariant, and it is necessary to investigate their behavior under the application of S. Treating $Z_1^\alpha Z_3^\beta Z_2^\gamma$ exactly as $Z_1^\alpha Z_2^\beta Z_3^\gamma$ was treated above we reach

$$F_2 = Z_4^\alpha [Z_3^\beta Z_2^\gamma + \varepsilon^{6\beta} Z_1^\beta Z_3^\gamma + Z_2^\beta Z_1^\gamma][1 + \varepsilon^{3(\beta+\gamma)} + \varepsilon^{6(\beta+\gamma)}],$$

which vanishes identically unless $\beta+\gamma \equiv o$. (mod. 3).

Similarly treating $Z_3^\alpha Z_2^\beta Z_1^\gamma$, we obtain a form which vanishes identically unless $\alpha+\beta \equiv o$ (mod 3). Hence we conclude that

$$\left.\begin{array}{l} \alpha+\beta \equiv o \text{ (mod 3)} \\ \beta+\gamma \equiv (o) \text{ (mod 3)} \end{array}\right\} \dots\dots\dots\dots \quad (2)$$

but $\alpha+\beta+\gamma \equiv o$ (mod 3).

$$\left.\begin{array}{l} \therefore\ \alpha \equiv o \quad \text{(mod 3)} \\ \quad\ \beta \equiv o \quad \text{(mod 3)} \\ \quad\ \gamma \equiv o \quad \text{(mod 3)} \end{array}\right\} \dots\dots\dots\dots \quad (3)$$

It is evident that if t be any three lettered term whose exponents satisfy the conditions (3) that $T^2 : (t) = (-1)^{\alpha+\beta+\gamma}\, t$.

$\therefore$ F_1, which is invariant under S, satisfies the relation

$$T^2 : F_1 = (-1)^{\alpha+\beta+\gamma}\, F_1$$

Hence $\alpha+\beta+\gamma \equiv o$ (mod 2) is a sufficient and necessary condition that F_1 is an absolute invariant under T^2. Similar remarks apply to F_2 and F_3. Hence we conclude

$\alpha+\beta+\gamma \equiv o$ (mod 2)

$\therefore$ from 1

$\alpha+\beta+\gamma \equiv o$ (mod 6)...................................... (4)

If, therefore, $Z_1^\alpha Z_2^\beta Z_3^\gamma$ is a term of an invariant form, then

$$\left.\begin{array}{l} \alpha \equiv \beta \equiv \gamma \equiv o \ \textit{(mod 3)} \\ \alpha+\beta+\gamma \equiv o \ \textit{(mod 6)} \end{array}\right\} \dots\dots\dots\dots \quad (5)$$

Applying now S and T to $Z_1^\alpha Z_2^\beta Z_3^\gamma$ and to the terms resulting until no new ones are reached, and forming the simplest symmetric functions, we obtain the form:

$$\begin{aligned}
&Z_1^\alpha Z_2^\beta Z_3^\gamma + Z_2^\alpha Z_3^\beta Z_1^\gamma + Z_3^\alpha Z_2^\gamma Z_1^\beta \\
&+(-1)^\alpha \left[Z_1^\alpha Z_3^\beta Z_4^\gamma + Z_2^\alpha Z_1^\beta Z_4^\gamma + Z_3^\alpha Z_2^\beta Z_4^\gamma \right] \\
&+(-1)^\beta \left[Z_4^\alpha Z_1^\beta Z_3^\gamma + Z_4^\alpha Z_2^\beta Z_1^\gamma + Z_4^\alpha Z_3^\beta Z_2^\gamma \right] \\
&+(-1)^\gamma \left[Z_3^\alpha Z_4^\beta Z_1^\gamma + Z_2^\gamma Z_4^\beta Z_1^\alpha + Z_2^\alpha Z_4^\beta Z_3^\gamma \right] = P_{\alpha,\beta,\gamma}, \text{ (say), (6).}
\end{aligned}$$

Giving to α, β, γ all possible values under limitations (5), we obtain a triply infinite system of invariant forms.

And this is the complete system, for we can only start with

$$Z_1^\alpha Z_2^\beta Z_3^\gamma,\ Z_1^\alpha Z_2^\beta Z_4^\gamma,\ Z_1^\alpha Z_3^\beta Z_4^\gamma \text{ or } Z_2^\alpha Z_3^\beta Z_4^\gamma.$$

But all these terms may be found in $P_{\alpha,\beta,\gamma}$ by a suitable interchange of α, β, γ. Certainly any rational function of the forms $P_{\alpha,\beta,\gamma}$ is also an invariant. Since $\alpha + \beta + \gamma \equiv o$ (2), at least *one* of the exponents is even.

Corollary 1. $P_{\alpha,\beta} = Z_1^\alpha Z_2^\beta + Z_2^\alpha Z_3^\beta + Z_3^\alpha Z_1^\beta + Z_3^\alpha Z_4^\beta + Z_1^\alpha Z_4^\beta + Z_2^\alpha Z_4^\beta$

$$(-1)^\alpha \left[Z_1^\alpha Z_3^\beta + Z_2^\alpha Z_1^\beta + Z_3^\alpha Z_2^\beta + Z_4^\alpha Z_1^\beta + Z_4^\alpha Z_2^\beta + Z_4^\alpha Z_3^\beta \right] \dots (6)$$

Cor. 2. $P_{\alpha,\beta} = -P_{\beta,\alpha};\ P_{\alpha,\alpha} = o.$

Cor. 3. If $\beta = o$ and $\gamma = o$, and $\alpha \equiv o$ (mod 2), we get a one-lettered form,

$P_\alpha = \sum Z_i^\alpha \ (i = 1 \text{--} 4).$

$\alpha \equiv o$ (mod 6).

2. To calculate the basis of our system.

*Hilbert has shown that if $\overline{Z}$ is a system of integral homogeneous forms of n variables that F, any form of $\overline{Z}$ can be represented by the expression

$F = \sum_{i=1}^{m} A_i F_i$ when m is finite and A_i are homogeneous integral functions of the variables and F_i are forms of $\overline{Z}$.

And in particular if $\overline{Z}$ be defined as a system of forms invariant under a group, that F any form of $\overline{Z}$ may be represented by a rational integral function of a finite number of forms of $\overline{Z}$. This finite number of forms is called the basis of the system.

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*See Mathematische Annalen, Vol. 36.

To find the basis of the forms of $P_{\alpha,\ \beta,\ \gamma}$ we need the recursion formulæ, which may be verified by computation.

$$P_{\alpha,\ \beta,\ \gamma} = P_{\beta,\ \gamma,\ \alpha} = P_{\gamma,\ \alpha,\ \beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

$$\Sigma\, Z_i^6 \cdot P_{\alpha,\ \beta,\ \gamma} - \Sigma\, Z_i^6 Z_k^6\, P_{\alpha - 6,\ \beta,\ \gamma} + \Sigma\, Z_i^6 Z_k^6 Z_l^6\, P_{\alpha - 12,\ \beta,\ \gamma}$$
$$- \Pi\, Z_i^6\, P_{\alpha - 18,\ \beta,\ \gamma} = P_{\alpha + 6,\ \beta,\ \gamma} \quad \ldots. (8)$$

$$\Sigma\, Z_i^6\, P_{\alpha,\ \beta,\ \gamma} - \Sigma\, Z_i^6 Z_k^6\, P_{\alpha,\ \beta - 6,\ \gamma} + \Sigma\, Z_i^6 Z_k^6 Z_l^6\, P_{\alpha,\ \beta,\ - 12,\ \gamma}$$
$$- \Pi\, Z_i^6\, P_{\alpha,\ \beta - 18,\ \gamma} = P_{\alpha,\ \beta + 6,\ \gamma} \quad \ldots.. (9)$$

$$\Sigma\, Z_i^6\, P_{\alpha,\ \beta,\ \gamma} - \Sigma\, Z_i^6 Z_k^6\, P_{\alpha,\ \beta,\ \gamma - 6} + \Sigma\, Z_i^6 Z_k^6 Z_l^6\, P_{\alpha,\ \beta,\ \gamma - 12}$$
$$- \Pi\, Z_i^6\, P_{\alpha,\ \beta,\ \gamma - 18} = P_{\alpha,\ \beta\ \gamma + 6} \quad \ldots. (10)$$

$$\Pi\, Z_i^6 \cdot \Sigma\, Z_i^6 Z_k^6\, P_{\alpha,\ \beta,\ \gamma} = \Pi\, X_i^{12} \Sigma\, X_i^6\, P_{\alpha - 6,\ \beta - 6,\ \gamma - 6} - \Pi\, Z_i^{18}$$
$$P_{\alpha - 12,\ \beta - 12,\ \gamma - 12} + \Sigma\, Z_i^6 Z_k^6 Z_l^6\, P_{\alpha + 6,\ \beta + 6,\ \gamma + 6} -$$
$$P_{\alpha + 12,\ \beta + 12,\ \gamma + 12}.$$

Making $\gamma = o$ in 8, 9, 10, we get recursion formulæ for $P\alpha,\ \beta$.

By a repeated and successive application of the formulæ 8, 9, 10 to any $P\alpha,\ \beta,\ \gamma$, it is expressed as a rational integral function of forms $P\alpha',\ \beta',\ \gamma'$ whose greatest index is 18, and therefore finite in number, and which is therefore the basis of the system.

I will add, however, that by a somewhat tedious reduction it can be shown that the system can be expressed as rational functions of

$$\Sigma\, Z_i^6,\ \Sigma\, Z_i^6 Z_k^6,\ \Sigma\, Z_i^6 Z_k^6 Z_l^6,\ Z_1 Z_2 Z_3 Z_4,\ P_{9,3},\ P_{15,3},\ P_{6,3,3},\ P_{6,9,3}.$$

Rate of Decrease of the Intensity of Sounds With Time of Propogation. By A. Wilmer Duff.

[Abstract.]

PART I.—THEORETICAL.

The intensity of sounds spreading in spherical waves from a source would, if no part of the energy of vibration were lost in the passage, vary inversely as the square of the distance. But it is certain that a considerable proportion of the sound energy must in every second be converted into heat, though no attempt seems to have been made to determine experimentally what proportion this is of the whole. The transformation of energy of vibration into heat energy takes place in three ways. In the