A Common Text-Book Error in the Theory of Envelopes.

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The cause of this communication is the recent appearance of several text-books on the calculus that embody an error in the theory of envelopes that dates at least as far back as Todhunter's calculus, and is now reproduced in all text-books under the impression, apparently, that it has acquired the sanction of authority, although Cayley pointed out the error nearly forty years ago, while the subject matter is presented in all text-books on Differential Equations in its correct form. The error consists in defining the envelope of a moving curve as the locus of its self-intersections, and then proving that the envelope touches the moving curve in every position—i. e., proving as true that which is often false—for the locus of self-intersections of a moving curve may cut the curve at any angle, as at right angles, wherever the two meet. A simple example is the curve $(y-m)^2=(x-3)^3$, whose locus of self-intersections, as m varies, is the straight line $x=3$, which cuts every curve of the given system at right angles. The fact is, that the envelope should be *defined* as the curve that touches every curve of a given system. It can then be shown it is *a* locus of self-intersections of the curves of the system, provided such self-intersections are not the singular points of the given system. The locus of such singular points is always a locus of self-intersections, but it is not in general an envelope of the system, and may cut every curve of the system at any constant or varying angle. The text-book blunder referred to is of the same logical character as would be the attempt to prove that a quadruped is a horse. To be sure, a horse is a quadruped, but not every quadruped is a horse. Thus a curve that touches every curve of a given system is a locus of self-intersections of the system, but not every locus of self-intersections of the system will touch every curve of the system. The error in the proof arises out of the assumption that if two points of a curve approach coincidence, the limiting position of the chord joining the two points is a tangent line at the point of coincidence. This is all right if the point of coincidence is not a singular point of the curve. But at a singular point, as a sharp point like the bottom of a letter V, the limiting position of two points that approach the point on opposite sides is absolutely indeterminate, and is not necessarily a tangent line at that point.