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Point-Invariants for the Lie Groups of the Plane.

By David A. Rothrock.

Among the many interesting and important applications of Lie's Theory of Transformation Groups none deserves more prominent mention than the application to invariant theory. Whether the invariants dealt with be functions or equations, surfaces and curves or points, equally interesting results are obtained. The present paper has to do with the determination of the point-invariants for the finite continuous groups of the plane as classified by Lie in Vol. XVI. of the Mathematische Annalen. In the first part of the paper is sketched a brief outline of the Lie theory leading up to the point-invariant, then follow the calculations of the invariant functions.

An infinitesimal point-transformation gives to x and y the increments

$$\delta x = \xi\,(x, y)\,\delta t,\ \delta y = \eta\,(x, y)\,\delta t,$$

respectively, where δt is an infinitesimal independent of x and y. Such infinitesimal transformations move a point x, y through a distance

$$\sqrt{\delta x^2 + \delta y^2} = \sqrt{\xi^2 + \eta^2} \cdot \delta t,$$

and in a direction given by

$$\delta y : \delta x = \eta : \xi.$$

The variation of any function ϕ (x, y) by this infinitesimal transformation is given by

$$\delta\phi = \frac{d\phi}{dx}\delta x + \frac{d\phi}{dy}\delta y = \left\{ \xi\,(x, y)\,\frac{d\phi}{dx} + \eta\,(x, y)\,\frac{d\phi}{dy} \right\} \delta t.*$$

The variation of a function f (x, y) may be taken as a definition of an infinitesimal transformation; in the Lie notation we have an infinitesimal transformation defined by

$$Xf \equiv \xi\,(x, y)\,\frac{df}{dx} + \eta\,(x, y)\,\frac{df}{dy}.$$

If a function ϕ (x, y) is to remain invariant by the operation X f, then the variation

$$\delta\,\varphi\,(x, y) = X\,\phi\,.\,\delta\,t$$

must vanish. Hence, a function ϕ (x, y) invariant by the infinitesimal transformation X f is determined as a solution of the linear partial differential equation

$$Xf = \xi\,(x, y)\,\frac{df}{dx} + \eta\,(x, y)\,\frac{df}{dy} = 0.$$

The infinitesimal transformation X f may be *extended* to include the increments of the co-ordinates of any number of points x_i, y_i, $(i = 1, 2 \ldots . n)$. We shall write this extended transformation thus:

$$Wf = \sum_1^n {}_i X^{(i)} f = \sum_1^n {}_i \left(\xi\,(x_i, y_i)\,\frac{df}{dx_i} + \eta\,(x_i, y_i)\,\frac{df}{dy_i} \right) \quad . \; . \; . \; . \; (2)$$

The functions of the co-ordinates of n points invariant by Wf will be the $2\,n - 1$ independent solutions of $Wf = 0$. n of these solutions may be selected in the form $\varphi\,(x_i, y_i)$, where ϕ (x, y) is a solution of $X\,f = 0$; the remaining n–1 solutions will in general differ from ϕ (x, y) in form.†

r infinitesimal transformations $X_1 f$, $X_2 f \ldots . X_r f$ are called *independent* when *no* relation of the form

$$c_1\,X_1\,f + c_2\,X_2\,f + \ldots\ldots + c_r X_r\,f = 0, \;(c_i = \text{const.}),$$

exists. If r independent infinitesimal transformations $X_k f$, $(k = 1 \ldots r)$, be so related as to form a group, then will

$$X_i\,(X_k f) - X_k\,(X_i f) = \sum_1^r {}_s\, c_{iks} X_s f, \;(c_{iks} = \text{constants}) \quad . \; . \; . \; (3)$$

*Throughout this paper $\frac{df}{dx}$, $\frac{df}{dy}$ are employed to denote *partial* differentials of f with respect to x and y.

† Lie: Theorie der Transformationsgruppen, Bd. I., § 59.

The transformations of the *r-parameter* group $X_k f$ may be extended according to the method of (2) above, giving

$$W_k f \equiv \sum_1^n i X_k^{(i)} f, \quad (k = 1 \ldots . r),$$

which determine the increments of a function f $(x_1, y_1; x_2, y_2; \ldots . x_n, y_n)$. Since the relations (3) exist for $X_i f$, $X_k f$, they must also exist for $W_i f$, $W_k f$, that is

$$W_i(W_k f) - W_k(W_i f) \equiv \sum_1^r s\, c_{iks}\, W_s f.$$

Hence, $W_1 f = 0$, $W_2 f = 0$, ... $W_r f = 0$ are known to form a *complete* system of linear partial differential equations in $2n$ variables x_i, y_i, with at least $2n - r$ independent solutions. These $2n - r$ solutions are the invariants of the co-ordinates of n points by the r-parameter group $X_k f$. These solutions we shall call *point-invariants.*

According to the method here outlined we shall determine the *point-invariants* of the finite continuous groups of the plane. In Lie's classification these groups are divided into two classes: (1) Inprimitive, or those groups which leave invariant *one* or *more* families of ∞' curves; (2) Primitive, or those groups leaving invariant *no* family of ∞' curves. Subdivisions of the imprimitive groups will be indicated in the text.

NOTE.—The results of the present paper were worked out early in the spring of 1898. Since that time there has appeared a short article by Dr. Lovette, June number, 1898, of Annals of Mathematics, upon the same subject. Only a few of the *projective* groups are considered, however. Among these are the special linear, and general linear groups.

SECTION I. INVARIANTS OF SUCH IMPRIMITIVE GROUPS AS LEAVE UNCHANGED MORE THAN ONE FAMILY OF ∞' CURVES.

The groups of this category have been reduced by Lie to such canonical forms that they leave invariant:

(A) ∞^∞ of families of ∞' curves: $\phi(x) + \psi(y) =$ constant,

(B) A single infinity of families of ∞' curves: $ax + by =$ coustant,

(C) Two families of ∞' curves: $x =$ constant, $y =$ constant.

(A) *The totality of curves* $\phi(x) + \psi(y) =$ constant *remains invariant.*

1. $\boxed{q}$.*

* Lie employs this symbol to enclose the members of a continuous group;

$$p = \frac{df}{dx}, \quad q = \frac{df}{dy}.$$

This is the only group of the class (A), and furnishes us when *extended* the linear partial differential equation

$$Wf = \sum_1^n {}_i \frac{df}{dy_i} = 0.$$

The invariants of the co-ordinates of n points by this group will be the $2n - 1$ independent solutions of $Wf = 0$, *i. e.*

$$x_i, \ \psi_j = y_1 - y_j, \ (i = 1 \ldots . n, \ j = 2 \ldots . n).$$

B. *All families of curves of the form* $ax + by =$ constant *remain invariant.*

2. $\boxed{p, q}$.

The complete system corresponding to this group is

$$W_1 f = \sum_1^n {}_i \frac{df}{dx_i} = 0, \quad W_2 f = \sum_1^n {}_i \frac{df}{dy_i} = 0,$$

with solutions

$$\phi_j = x_1 - x_j, \ \psi_j = y_1 - y_j, \ (j = 2 \ldots . n).$$

The functions ϕ, ψ are the required invariants.

3. $\boxed{q, xp + yq}$.

From this group we have

$$W_1 f = \sum_1^n {}_i \frac{df}{dy_i} = 0, \quad W_2 f = \sum_1^n {}_i \left\{ x_i \frac{df}{dx_i} + y_i \frac{df}{dy_i} \right\} = 0.$$

These two linear partial differential equations evidently have as solutions

$$\xi_j = \frac{x_j}{x_1}, \ u_k = \frac{y_1 - y_k}{y_1 - y_2}, \ \sigma = \frac{y_1 - y_2}{x_1}, \ (j = 2 \ldots n, \ k = 3 \ldots n),$$

which are the invariants sought.

4. $\boxed{p, q, xp + yx}$.

This three-parameter group furnishes us the complete system

$$\sum_1^n {}_i \frac{df}{dx_i} = \sum_1^n {}_i \frac{df}{dy_i} = \sum_1^n {}_i \left\{ x_i \frac{df}{dx_i} + y_i \frac{df}{dy_i} \right\} = 0.$$

The first two of these equations have solutions

$$\phi_j = x_1 - x_j, \ \psi_j = y_1 - y_j, \ (j = 2 \ldots n),$$

which as new variables reduce the last equation to the form

$$\sum_2^n {}_j \left\{ \phi_j \frac{df}{d\phi_j} + \psi_j \frac{df}{d\psi_j} \right\} = 0.$$

Hence, the invariants are

$$U_k = \frac{\phi_k}{\phi_2} = \frac{x_1 - x_k}{x_1 - x_2}, \ V_k = \frac{\psi_k}{\psi_2} = \frac{y_1 - y_k}{y_1 - y_2}, \ \sigma = \frac{y_1 - y_2}{x_1 - x_2}, \ (k = 3 \ldots n).$$

(C.) *The families of curves* $x =$ *constant*, $y =$ *constant*, *remain invariant.*

5. $|q, yq|$.

The complete system corresponding to this group,

$$\sum_1^n{}_i \frac{df}{dy_i} = \sum_1^n{}_i \; y_i \frac{df}{dy_i} = 0,$$

has as solutions

x_i, and $\psi_k = (y_1 - y_k) : (y_1 - y_2)$, $(i = 1 \ldots\ldots n,\; k = 3 \ldots\ldots n)$.

Hence. x_i and ψ_k are the invariants.

6. $|q, yq, y^2q|$.

This is the general projective group in one variable, and leaves invariant x_i and the cross-ratios of any four ordinates.

$$\sum_1^n{}_i \frac{df}{dy_i} = \sum_1^n{}_i \; y_i \frac{df}{dy_i} = \sum_1^n{}_i \; y_i^2 \frac{df}{dy_i} = 0.$$

The first two equations of this system have solutions x_i, ψ_k of 5 above. Introducing these solutions as new variables in the last equation, we have

$$\sum_3^n{}_k \; \psi_k \, (\psi_k - 1) \frac{df}{d\psi_k} = 0,$$

whose solutions are

$$x_i, \; \xi_l = \frac{\psi_l - 1}{\psi_l} : \frac{\psi_3 - 1}{\psi_3} = \frac{y_2 - y_l}{y_2 - y_3} : \frac{y_1 - y_l}{y_1 - y_3}, \; (l = 4 \ldots\ldots n).$$

7. $|q, yq, p|$.

This group leaves invariant

$\psi_k = (y_1 - y_k) : (y_1 - y_2)$, and $\phi_j = x_1 - x_j$, $(k = 3 \ldots\ldots n,\; j = 2 \ldots\ldots n)$.

8. $|q, yq, y^2q, p|$.

The invariants of this group are clearly

$$\xi_l = \frac{y_2 - y_l}{y_2 - y_3} : \frac{y_1 - y_l}{y_1 - y_3}, \text{ as in 6, and}$$

$$\phi_j = x_1 - x_j, \text{ as in 7.}$$

9. $|q, p, xp + cyq|$.

The solutions $\psi_j = y_1 - y_j$, $\phi_j = x_1 - x_j$ of the first two equations obtained from this group, when introduced in the last one, give

$$\sum_2^n{}_j \left\{ \phi_j \frac{df}{d\phi_j} + c\, \psi_j \frac{df}{d\psi_j} \right\} = 0.$$

The required invariants of the group may now be chosen as

$$U_k = \frac{x_1 - x_k}{x_1 - x_2}, \; V_k = \frac{y_1 - y_k}{y_1 - y_2}, \; \sigma = \frac{(x_1 - x_2)^c}{y_1 - y_2}, (k = 3 \; \ldots . \; n).$$

10. $\vert\; q, \; yq, \; p, \; xp \;\vert$.

Comparing this group with 7, we have at once the invariants

$$U_k = \frac{x_1 - x_k}{x_1 - x_2}, \; V_k = \frac{y_1 - y_k}{y_1 - y_2}, \quad (k = 3 \ldots n).$$

11. $\vert\; q, \; yq, \; y^2q, \; p, \; xp \;\vert$.

By comparison with 8 and 10, it will be seen that this five-parameter group leaves invariant

$$\xi_l = \frac{y_2 - y_l}{y_1 - y_l} : \frac{y_2 - y_3}{y_1 - y_3}, \quad U_k = \frac{x_1 - x_k}{x_1 - x_2}, \; (l = 4 \ldots n, \; k = 3, \ldots n).$$

12. $\vert\; q, \; yq, \; y^2q, \; p, \; xp, \; x^2p \;\vert$.

Comparing with 6, it will be seen that this group leaves invariant the cross-ratios of any four abscissas, and ordinates:

$$\xi_l = \frac{y_2 - y_l}{y_2 - y_3} : \frac{y_1 - y_l}{y_1 - y_3}, \quad \sigma_l = \frac{x_2 - x_l}{x_2 - x_3} : \frac{x_1 - x_l}{x_1 - x_3}, \quad (l = 4 \; .. \; n).$$

13. $\vert\; p + q, \; xp + yq, \; x^2p + y^2q \;\vert$.

This group furnishes the complete system

$$\sum_1^n {}_i \left\{ \frac{df}{dx_i} + \frac{df}{dy_i} \right\} = \sum_1^n {}_i \left\{ x_i \frac{df}{dx_i} + y_i \frac{df}{dy_i} \right\} = \sum_1^n {}_i \left\{ x_i^2 \frac{df}{dx_i} + y_i^2 \frac{df}{dy_i} \right\} = 0.$$

Selecting

$$\varphi_j = x_1 - x_j, \; \psi_j = y_1 - y_j, \; \sigma = x_1 - y_1$$

as solutions of the first equation, we have the remaining equations in the form

$$W_1 f = \sum_2^n {}_j \left\{ \varphi_j \frac{df}{d\phi_j} + \psi_j \frac{df}{d\psi_j} \right\} + \sigma \frac{df}{d\sigma} = 0,$$

$$W_2 f = \sum_2^n {}_j \left\{ \phi_j^2 \frac{df}{d\phi_j} + \psi_j^2 \frac{df}{d\psi_j} \right\} + \sigma^2 \frac{df}{d\sigma} = 0.$$

Solutions of $W_1 f = 0$ may be taken in the form

$$u_k = \frac{\phi_k}{\varphi_2}, \; v_k = \frac{\psi_k}{\psi_2}, \; \omega_1 = \frac{\phi_2}{\sigma}, \; \omega_2 = \frac{\psi_2}{\sigma}.$$

Expressing $W_2 f = 0$ in terms of these new variables,

$$\sum_3^n {}_k \left\{ u_k (1 - u_k) \frac{df}{du_k} + v_k (1 - v_k) \frac{df}{dv_k} \right\} + \omega_1 (1 - \omega_1) \frac{df}{d\omega_1} + \omega_2 (1 - \omega_2) \frac{df}{d\omega_2} = 0.$$

We may choose the solutions of this last equation as the cross-ratios

$$r_l = \frac{u_l\,(1-u_3)}{u_3\,(1-u_l)} = \frac{x_2-x_1}{x_2-x_3} : \frac{x_l-x_1}{x_l-x_3},\quad \sigma_l = \frac{v_l\,(1-v_3)}{v_3\,(1-v_l)} = \frac{y_2-y_1}{y_2-y_3} : \frac{y_l-y_1}{y_l-y_3},$$

$$(l = 4, \ldots\ldots n),$$

and the ratios

$$t_1 = \frac{\omega_1\,(1-\omega_2)}{\omega_2\,(1-\omega_1)} = \frac{x_1-x_2}{y_1-y_2} : \frac{y_1-x_2}{x_1-y_2},$$

$$t_2 = \frac{u_3\,(1-v_3)}{v_3\,(1-u_3)} = \frac{x_1-x_3}{x_2-x_3} : \frac{y_1-y_3}{y_2-y_3},$$

$$t_3 = \frac{u_3\,(1-\omega_1)}{\omega_1\,(1-u_3)} = \frac{x_1-x_3}{x_2-x_3} : \frac{x_1-x_2}{y_1-x_2}.$$

SECTION II. INVARIANTS OF SUCH IMPRIMITIVE GROUPS AS LEAVE UNCHANGED ONE FAMILY OF ∞' CURVES.

The remaining groups of the imprimitive type leave invariant one family of ∞' curves, and have been reduced by Lie* to such canonical forms that this invariant family is $x = \text{const.}$

14. $\left|\overline{\underline{X_1 q,\ X_2 q,\ X_3 q,\ \ldots\ldots\ X_r q}}\right|.$

In this group X_k is a function of x alone, and $r > 1$. Each curve of the family $x = \text{const.}$ remains singly invariant.

The complete system of linear partial differential equations

$$W_k f \equiv \sum_1^n{}_i\ X_k(x_i) . \frac{df}{dy_i} = 0,\ (k = 1\ \ldots .\ r),$$

corresponding to this group, has as solutions x_1, x_2, ... x_n and $n - r$ other independent functions D_s, $(s = 1, 2\ \ldots\ n - r)$, which we shall define as the $n - r$ determinants of the matrix

$$\left\|\begin{array}{ccccc} y_1 & y_2 & \cdots & y_{r+s} & \cdots\ y_n \\ X_{1(x_1)} & X_{1(x_2)} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{r(x_1)} & X_{r(x_2)} & \cdots & \cdots & \cdots \end{array}\right\|, \qquad (M_r)$$

$$n \overline{>} r + 1$$

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* Lie; Math. Annalen, Bd. XVI; Contin. Gruppen, Chap. 13.

formed by filling the $(r+1)$-th column successively by the $(r+1)$, the $(r+2)$, ..., the n-th column. The invariants are clearly x_i and D_s, $(i = 1 \ldots . n,$ $s = 1, \ldots . n - r)$.

15. $$\boxed{\begin{matrix} X_1q, X_2q, X_3q, \ldots . X_{r-1}q, yq \\ r > 2 \end{matrix}}.$$

This group furnishes the complete system

$$W_k f = \sum_1^n {}_i \, X_k(x_i) \cdot \frac{df}{dy_i} = 0, \; Yf = \sum_1^n {}_i \, y_i \cdot \frac{df}{dy_i} = 0.$$

The solutions of $W_k f = 0$ are clearly x_i and the determinants D_s, $(s = 0, 1 \ldots$ $n - r)$, of a matrix (M_{r-1}) constructed similar to (M_r) in 14. $Yf = 0$ requires the ratios $y_i : y_k$ to appear in the final solutions. Hence, we may write as invariants x_i and

$$\xi_t = D_t : D_0, \; (t = 1 \ldots . n - r).$$

16. $$\boxed{\begin{matrix} e^{\alpha_k x}q, \; xe^{\alpha_k x}q, \; x^2e^{\alpha_k x}q, \; \ldots . \; x^{\zeta_k}e^{\alpha_k x}q, \; p \\ k = 1, \ldots . m, \; \sum_1^m {}_k \, \zeta_k + m = r - 1, \; r > 2 \end{matrix}}.$$

From this group we obtain

$$W_k^{t_k} f = \sum_1^n {}_i \, (x_i)^{t_k} \, e^{\alpha_k x_i} . \frac{df}{dy_i} = 0, \;\; Xf = \sum_1^n {}_i \, \frac{df}{dx_i} = 0, \; (t_k = 0, 1, \ldots . \zeta_k).$$

The solutions $\varphi_j = x_j - x_1$ of the last equation are also solutions of the system. By dividing the remaining equations, respectively, by $e^{\alpha_k x_1}$, the exponents of e become functions of φ_j. The independent determinants D_s, $(s = 0,$ $1 \ldots n - r)$, of the matrix (M_{r-1}), formed as indicated in 15, will be solutions. The invariants are, therefore, φ_j and D_s.

17. $$\boxed{\begin{matrix} e^{\alpha_k x}q, \; xe^{\alpha_k x}q, \; x^2e^{\alpha_k x}q, \; \ldots . \; x^{\zeta_k}e^{\alpha_k x}q, \; yq, \; p \\ k = 1 \ldots . m, \; \sum_1^m {}_k \, \zeta_k + m = r - 2, \; r > 3 \end{matrix}}.$$

The complete system given by this group is

$$W_k^{t_k} f = \sum_1^n {}_i \, (x_i)^{t_k} . e^{\alpha_k x_i} . \frac{df}{dy_i} = 0, \; Yf = \sum_1^n {}_i \, y_i \frac{df}{dy_i} = 0, \; Xf = \sum_1^n {}_i \, \frac{df}{dx_i} = 0,$$

$$(t_k = 0, 1 \ldots . \zeta_k).$$

As in 16, the functions $\phi_j = x_j - x_1$ are solutions of $Xf = 0$ and of the system. If a matrix be constructed as indicated in 14 and 16, from the coefficients of the first $r-2$ equations, it will be observed that the independent determinants D_s, $(s = -1, 0, 1 \ldots . n-r)$, will be linear and homogeneous in y_i with coefficients composed of functions of ϕ_j. D_s will then be solutions of all equations except $Yf = 0$, which requires the ratios of y_i to appear. Hence, the invariants may be written

$$\phi_j = x_j - x_1,\ \zeta_t = D_t : D_{-1},\ (j = 2 \ldots . n,\ t = 0, 1 \ldots . n - r).$$

18. $\boxed{q,\ xq,\ x^2q,\ \ldots .\ x^{r-3}q.\ p,\ xp + cyq \quad r > 3}$.

Here the complete system is

$$W_k f \equiv \sum_1^n {}_i\, x_i^k \frac{df}{dy_i} = 0,\ (k = 0, 1 \ldots . r-3),\ Xf \equiv \sum_1^n {}_i \frac{df}{dx_i} = 0,$$

$$Yf \equiv \sum_1^n {}_i \left\{ x_i \frac{df}{dx_i} + c\, y_i \frac{df}{dy_i} \right\} = 0.$$

The solutions of $W_0 f = 0$, $Xf = 0$ are

$$\psi_j = y_1 - y_j,\ \phi_j = x_1 - x_j$$

Yf expressed in terms of ψ, ϕ becomes

$$Y_1 f \equiv \sum_2^n {}_j \left\{ \phi_j \frac{df}{d\phi_1} + c\psi_1 \frac{df}{\phi\psi_j} \right\} = 0,$$

with solutions

$$u_k = \phi_k : \phi_2,\ v_j = \psi_j : (\phi_j)^c,\ (k = 3 \ldots n).$$

The functions u_k are solutions of the system. We find on introducing u_k and v_j as new variables in Wf, the partial differential equations

$$W_t' f \equiv \frac{df}{dv_2} - \sum_3^n {}_k\, u_k^{t-c}.\ \frac{df}{dv_k} = 0,\ (t = 1, 2, \ldots . r-3),$$

whose solutions may be expressed as determinants D_s of the matrix.

$$\left\| \begin{matrix} v_2 & v_3 & v_4 \ldots\ldots v_{r-3+s} \ldots\ldots v_n \\ 1 & u_3^{1-c} & u_4^{1-c} \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ 1 & u_3^{r-3-c} & u_4^{r-3-c} \ldots\ldots\ldots\ldots u_n^{r-3-c} \end{matrix} \right\|.$$

Hence, the point-invariants are

$$u_k = \frac{x_1 - x_k}{x_1 - x_2}, \quad D_s, \ (k = 3 \ldots . n, \ s = 1, \ldots . n - r + 2).$$

19. $\boxed{q, xq, x^2q, \ldots . x^{r-3}q, p, xp + [(r-2)y + x^{r-2}]q \atop r > 2}$.

The solutions of the complete system

$$W_kf \equiv \sum_1^n{}_i \, x_i^k \frac{df}{dy_i} = 0, \ (k = 0, 1 \ldots r - 3), \quad Xf \equiv \sum_1^n{}_i \frac{df}{dx_i} = 0,$$

$$Yf \equiv \sum_1^n{}_i \left\{ x_i \frac{df}{dx_i} + [(r-2)y_i + x_i^{r-2}] \frac{df}{dy_i} \right\} = 0,$$

may be obtained in a manner similar to 18. The solutions ϕ_j, ψ_j of $Xf = 0$, $Wf = 0$, introduced as new variables in Yf, give

$$Y'f = \sum_2^n{}_j \left\{ \phi_j \frac{df}{d\phi_j} + [(r-2)\psi_j + \phi_j^{r-2}]. \frac{df}{d\psi_j} \right\} = 0,$$

with solutions

$$u_k = \phi_k : \phi_2, \ v_j = \log \phi_j - \frac{\psi_j}{\phi_j^{r-2}}, \ (k = 3 \ldots \ldots n, \ j = 2, \ldots n).$$

Introducing u_k, v_j as new variables in Wf, and reducing, we find

$$W_tf = \frac{df}{dv_2} + \sum_2^n{}_k \, u_k^{(t+2-r)} . \frac{df}{dv_k} = 0, \ (t = 1, \ldots r - 3),$$

whose solutions are u_k and the determinants D_t of a matrix constructed as in 18.

The invariants are, therefore,

$$u_k = \frac{x_1 - x_k}{x_1 - x_2}, \quad D_s, \ (k = 3 \ldots n, \ s = 1, \ldots n - r + 2).$$

20. $\boxed{q, xq, x^2q, \ldots \ldots x^{r-4}q, yq, p, xp \atop r > 3}$.

For this group

$$W_tf \equiv \sum_1^n{}_i \, x_i^t \frac{df}{dy_i} = 0, \ t = 0, 1, \ldots r - 4),$$

$$Yf \equiv \sum_1^n{}_i \, y_i \frac{df}{dy_i} = 0, \quad X_1f \equiv \sum_1^n{}_i \frac{df}{dx_i} = 0, \quad X_2f \equiv \sum_1^n{}_i \, x_i \frac{df}{dx_i} = 0.$$

The last two equations show that the ratios of the differences of the x's, say

$$u_k = \frac{x_1 - x_k}{x_1 - x_2}, \ (k = 3, \ldots . n),$$

shall appear in the final solutions. The $n-r+3$ independent determinants D_s, $(s=0, 1 \ldots n-r+2)$, of the matrix

$$\begin{vmatrix} y_1 & y_2 & y_3 \;\cdots\cdots & y_{r-2+s} \cdots\cdots & y_n \\ 1 & 1 & 1 \;\cdots\cdots & 1 \cdots\cdots & 1 \\ x_1 & x_2 & x_3 \;\cdots\cdots & x_{r-2+s} \cdots\cdots & x_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_1^{r-4} & x_2^{r-4} & x_3^{r-4} \;\cdots & x_{r-2+s}^{r-4} \cdots & x_n^{r-4} \end{vmatrix}$$

are solutions of the first $r-3$ equations $W_k f=0$. These determinants are, at the same time, homogeneous in y_i and in x_i-x_k; their ratios will, therefore, satisfy the requirements of u_k and $Yf=0$. Hence, we may write our $2n-r$ invariants as $u_k=(x_1-x_k):(x_1-x_2)$ and $R_t=D_t:D_o$, $(k=3 \ldots n, t=1 \ldots n-r+2)$.

21. $\boxed{q, xq, x^2q, \ldots x^{r-4}q, p, 2xp+(r-4)yq, x^2p+(r-4)xyq \quad r>4}$.

From this group we obtain the differential equations

$$W_tf=\sum_1^n{}_i\, x_i^t \frac{df}{dy_i}=0, \ (t=0, 1 \ldots r-4), \ Xf=\sum_1^n{}_i \frac{df}{dx_i}=0.$$

$$X_1f=\sum_1^n{}_i \left\{2x_i \frac{df}{dx_i}+(r-4)\, y_i \frac{df}{dy_i}\right\}=0,$$

$$X_2f=\sum_1^n{}_i \left\{x_i^2 \frac{df}{dx_i}+(r-4)\, x_i\, y_i \frac{df}{dy_i}\right\}=0.$$

The solutions of $W_0f=0$, $Xf=0$ are $\psi_j=y_1-y_j$, $\varphi_j=x_1-x_j$, respectively. X_2f when expressed in these new variables becomes

$$X_2'f=\sum_2^n{}_j \left\{\phi_j^2 \frac{df}{d\phi_j}+(r-4)\, \phi_j\, \psi_j \frac{df}{d\psi_j}\right\}=0,$$

whose solutions may be selected in the forms

$$u_k=\frac{1}{\varphi_2}-\frac{1}{\varphi_k}, \ v_j=\frac{\psi_j}{\varphi_j^{r-4}}, \ (k=3 \ldots n, \ j=2 \ldots n).$$

$$X_1'f=2\sum_3^n{}_k\, u_k \frac{df}{du_k}+(r-4)\sum_2^n{}_j\, v_j \frac{df}{dv_j}=0$$

has solutions

$\sigma_l=u_l:u_3$, $\zeta_k=v_k:u_k^{\frac{1}{2}(r-4)}$, $\zeta_2=v_2:u_3^{\frac{1}{2}(r-4)}$, $(k=3 \ldots n, \ l=4 \ldots n)$.

The remaining equations $W\,f$ may be expressed in terms of σ, ζ in the following forms:

$$\frac{df}{d\zeta_2}+\frac{df}{d\zeta_3}+\sum_4^n l\ (\sigma_l)^{-d}\cdot\frac{df}{d\zeta_l}=0,\ [d=\tfrac{1}{2}\,(r-4)],$$

$$W_t'\,f\equiv\frac{df}{d\zeta_3}+\sum_4^n l\ (\sigma)^{\,t-d}\cdot\frac{df}{d\zeta_l}=0,\ (t=1,\ \dots.\ r-5).$$

The solutions of these equations may be expressed as the determinants D_s, $(s=1,\ \dots.\ n-r+3)$, of the matrix

$$\left\|\begin{matrix} \zeta_2 & \zeta_3 & \zeta_4 & \dots\dots & \zeta_{r-3+s} & \dots\dots & \zeta_n \\ 1 & 1 & \sigma_4^{-d} & \dots\dots & \sigma_{r-3+s}^{-d} & \dots\dots & \sigma_n^{-d} \\ 0 & 1 & \sigma_4^{1-d} & \dots\dots & \dots\dots & \dots\dots & \dots \\ \dots & \dots & \dots & \dots\dots & \dots\dots & \dots\dots & \dots \\ \dots & \dots & \dots & \dots\dots & \dots\dots & \dots\dots & \dots \\ 0 & 1 & \sigma_4^{r-5-d} & \dots\dots & \dots\dots & \dots\dots & \dots \end{matrix}\right\|.$$

The required invariants are, therefore,

$$\sigma_l=\frac{x_2-x_l}{x_1-x_l}:\frac{x_2-x_3}{x_1-x_3},\ D_s,\ (l=4\ \dots\ n,\ s=1,\ \dots.\ n-r+3).$$

22. $\boxed{q,\ xq,\ x^2q,\ \dots\dots\ x^{r-5}q,\ yq,\ p,\ xp,\ x^2p+(r-5)\,xyq}$.
$r-5$

This group furnishes the system

$$W_tf\equiv\sum_1^n i\ x_i^t\,\frac{df}{dy_i}=0,\ (t=0,1\ \dots.\ r-5),\quad Yf\equiv\sum_1^n i\ y_i\,\frac{df}{dy_i}=0,$$

$$Xf\equiv\sum_1^n i\,\frac{df}{dx_i}=0,\quad X_1f\equiv\sum_1^n i\ x_i\,\frac{df}{dx_i}=0,$$

$$X_2f\equiv\sum_1^n i\left\{x_i^2\,\frac{df}{dx_i}+(r-5)\,x_iy_i\,\frac{df}{dy_i}\right\}=0.$$

The solutions of $W_0f=0$ are $\psi_j=y_1-y_j$, those of $X_1f=0$, $Xf=0$ are $u_k=(x_1-x_k):(x_1-x_2)$. X_2f expressed in ψ, u is

$$X_2^1f\equiv\sum_3^n k\left\{u_k\,(u_k-1)\,\frac{df}{du_k}+(r-5)\,u_k\cdot\frac{df}{d\psi_k}\right\}+(r-5)\,\psi_2\,\frac{df}{d\psi_2}=0,$$

whose solutions are

$$\sigma_l=\frac{u_3\,(u_l-1)}{u_l\,(u_3-1)},\ \zeta_k=\frac{\psi_k}{(u_k-1)^{r-5}},\ \zeta_2=\psi_2\left(\frac{u_3}{u_3-1}\right)^{r-5},\ (l=4\ \dots\ n,\ k=3\ \dots\ n).$$

The remaining equations expressed in these new variables are

$$\left.\begin{aligned} &\frac{df}{d\zeta_2}+\frac{df}{d\zeta_3}+\sum_4^n l\,(\sigma_l)^{5-r}.\ \frac{df}{d\zeta_l}=0,\\ &W_t^1 f\equiv\frac{df}{d\zeta_3}+\sum_4^n l\,(\sigma_l)^{-t}.\ \frac{df}{d\zeta_l}=0,\ (t=1\ \ldots\ r-5),\end{aligned}\right\}\ (1)$$

$$Y^1 f\equiv\sum_2^n j\ \zeta_j.\ \frac{df}{d\zeta_j}=0.$$

The determinants D_s ($s=0, 1 \ldots n-r+3$), of the matrix formed from equations (1) as in 21 will be solutions of (1). D_s will be linear in ζ, but $Y^1 f=0$ requires the ratios of ζ's. We may write our invariants as

$$\sigma_l=\frac{x_2-x_l}{x_1-x_l}:\frac{x_2-x_3}{x_1-x_3},\ R_t=D_t:D_0,\ (l=4\ \ldots\ n,\ t=1\ \ldots\ n-r+3).$$

23. $|\ p,\ 2xp+yq,\ x^2p+xyq.\ |$.

This projective group, leaving invariant the x-axis, furnishes us the complete system

$$\sum_1^n i\ \frac{df}{dx_i}=\sum_1^n i\left\{2x_i\frac{df}{dx_i}+y_i\frac{df}{dy_i}\right\}=\sum_1^n i\left\{x_i^2\frac{df}{dx_i}+x_iy_i\frac{df}{dy_i}\right\}=0.$$

The first of these equations has solutions y_i and $\phi_j=x_1-x_j$. The last equation then becomes

$$\sum_2^n j\left\{\phi_j^2\frac{df}{d\phi_j}+y_j\phi_j\frac{df}{dy_j}\right\}=0,$$

with solutions

$$u_k=\frac{1}{\phi_2}-\frac{1}{\phi_k},\ v_j=\frac{\phi_j}{y_j},\ y_1.$$

The second equation is now

$$y_1\frac{df}{dy_1}+\sum_2^n j\ v_j\frac{df}{dv_j}-2\sum_3^n k\ u_k\frac{df}{du_k}=0,$$

whose solutions we may choose in the forms

$$\delta_l=\frac{u_l}{u_3},\ \zeta_k=\frac{v_k}{v_2},\ \zeta_2=\frac{v_2}{y_1},\ \zeta_1=y_1v_2u_3,\ (l=4\ \ldots\ n,\ k=3\ \ldots\ n).$$

Our invariants are

$$\delta_l=\frac{x_2-x_l}{x_2-x_3}:\frac{x_1-x_l}{x_1-x_3},\ \zeta_k=\frac{x_1-x_k}{x_1-x_2}:\frac{y_k}{y_2},\ \zeta_2=\frac{{}_1x-x_2}{y_1y_2},\ \zeta_1=\frac{x_2-x_3}{x_1-x_3}:\frac{y_2}{y_1},$$

whose geometric significance is apparent.

24. $|\ yq,\ p,\ xp,\ x^2p+nyq\ |$.

This four-parameter projective group yields the complete system

$$\sum_1^n{}_i\, y_i\frac{df}{dy_i} = \sum_1^n{}_i\, \frac{df}{dx_i} = \sum_1^n{}_i\, x_i\,\frac{df}{dx_i} = \sum_1^n{}_i\, \left\{x_i^2\frac{df}{dx_i} + x_i y_i\,\frac{df}{dy_i}\right\} = 0.$$

Here, we introduce the solutions

$$\phi_j = x_1 - x_j,\ \psi_j = \frac{y_j}{y_1},$$

of the first two equations in the last two, and have

$$\sum_2^n{}_j\, \phi_j\frac{df}{\sigma\phi_j} = \sum_2^n{}_j\, \left\{\phi_j^2\,\frac{df}{\sigma\phi_j} + \phi_j\,\psi_j\,\frac{df}{\sigma\psi_j}\right\} = 0.$$

The last of these new equations is satisfied by

$$u_k = \frac{1}{\varphi_1} - \frac{1}{\varphi_k},\ v_j = \frac{\varphi_j}{\psi_j},\ (k = 3\ \ldots.\ n,\ j = 2\ \ldots.\ n);$$

the first now becomes

$$\sum_3^n{}_k\, u_k\,\frac{df}{du_k} - \sum_2^n{}_j\, v_j\,\frac{df}{dv_j} = 0.$$

The solutions of this equation are the requred invariants:

$$\sigma_l = \frac{u_l}{u_3} = \frac{x_2 - x_l}{x_2 - x_3} : \frac{x_1 - x_l}{x_1 - x_3},\ (l = 4\ \ldots.\ n),$$

$$\tau_k = \frac{v_k}{v_2} = \frac{x_1 - x_k}{x_1 - x_2} : \frac{y_k}{y_2},\ (k = 3\ \ldots.\ n),$$

$$\omega = u_3\,v_2 = \frac{x_2 - x_3}{x_1 - x_3} : \frac{y_2}{y_1}.$$

SECTION III. INVARIANTS OF THE PRIMITIVE GROUPS.

The remaining finite continuous groups of the plane leave *no* family of ∞' curves invariant, and may be reduced by a proper choice of variables to some one of the canonical forms known as (1) special linear, (2) general linear, (3) general projective.*

25. *The special linear group*

$$\overline{p,\ q,\ xq,\ xp - yq,\ yp}\ .$$

The invariant functions of the coördinates of n points will be the $2n - 5$ independent solutions of the complete system

$$\sum_1^n{}_j\,\frac{df}{dx_i} = \sum_1^n{}_i\,\frac{df}{dy_i} = \sum_1^n{}_i\, x_i\,\frac{df}{dy_i} = \sum_1^n{}_i\,\left\{x_i\frac{df}{dx_i} - y_i\,\frac{df}{dy_i}\right\} = \sum_1^n{}_i\, y_i\,\frac{df}{dx_i} = 0. \quad (1)$$

*Lie: Math. Annalen, Bd. XVI, p. p. 518-522, also Contin. Gruppen, p. 351.

The first two equations show the solutions of the system to be functions of

$$\varphi_j = x_1 - x_j,\ \psi_j = y_1 - y_j,\ (j = 2 \ldots . n).$$

The remaining equations then take the forms

$$\sum_2^n j\ \phi_j \frac{df}{d\psi_j} = \sum_2^n j \left\{ \phi_j \frac{df}{d\varphi_j} - \psi \frac{df}{d\psi_j} \right\} = \sum_2^n j\ \psi_j \frac{df}{d\varphi_j} = 0. \quad \ldots\ldots\ldots (2)$$

The second of these equations has solutions

$$u_j = \varphi_j \psi_j,\ v_k = \phi_2 \psi_k,\ (k = 3 \ldots . n).$$

With u and v as new variables, the first and third of equations (2) become

$$\frac{df}{du_2} + \sum_3^n k \left\{ \frac{u_k^2}{v_k^2} \frac{df}{du_k} + \frac{u_k}{v_k} \frac{df}{dv_k} \right\} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$u_2 \frac{df}{du_2} + \frac{1}{u_2} \sum_3^n k\ v_k^2 \frac{df}{u_k} + \frac{1}{v_3} \sum_3^n k\ v_k \frac{df}{dv_k} = 0. \quad \ldots\ldots (4)$$

The solutions of (3) are found to be

$$\sigma_k = \frac{v_k}{u_k},\ \zeta_k = u_2 - \frac{v_k^2}{u_k}.$$

Equation (4) then reduces to

$$\sum_3^n k \left\{ \sigma_k \zeta_k \frac{df}{d\sigma_k} + \zeta_k^2 \frac{df}{d\zeta_k} \right\} = 0,$$

whose solutions may be written

$$I'_k = \frac{\zeta_k}{\sigma_k},\ J_l = \frac{1}{\zeta_l} - \frac{1}{\zeta_3},\ (k = \ldots . n,\ l = 4 \ldots . n).$$

Since any functions of I', J will be the solutions of (1), we may choose

$$I'_k = \frac{\zeta_k}{\sigma_k} = \left| 1\ 2\ k \right| \text{ and } D_l = J_l I_3' I_l' = \left| 1\ 3\ l \right|,$$

$$\text{where } \left| i\ j\ k \right| = \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix},$$

as solutions, and, therefore, as the $2n - 5$ invariants of the group.

The forms of I' and D show that *the special linear group leaves invariant all areas.*

26. *The general linear group*

$$\left| \overline{p,\ q,\ xq,\ xp - yq,\ yp,\ xp + yq} \right. .$$

This group furnishes a complete system of six linear partial differential equations, the first five of which are identical with equations (1) of the preceding section. Hence we need only determine the functions of I' and D which satisfy

$$\sum_1^n i \left\{ x_i \frac{df}{dx_i} + y_i \frac{df}{dy_i} \right\} = 0.$$

This equation requires x, y to enter in the final solutions to the degree zero. Hence, we may write at once the invariants in the form

$$I_l = \frac{I_l'}{I_3'} = |\,1\,2\,l\,| : |\,1\,2\,3\,|,$$

$$J_l = \frac{D_l}{I_3} = |\,1\,3\,l\,| : |\,1\,2\,3\,|,\ (l = 4 \ldots . n).$$

I and J show that by *the general linear group the ratio of areas remains constant.*

27. *The general proejctive group*

$$\overline{\underline{p,\ q,\ xq,\ xp - yq,\ yp,\ xp + yq,\ x^2p + xyq,\ xyp + y^2q}}\ .$$

The members of this group extended and equated to zero furnish a complete system of eight linear partial differential equations, the first six of which are identical with those of the general linear group, and therefore have solutions I, J, defined in 26. The last two equations,

$$\sum_1^n i \left\{ x_i^2 \frac{df}{dx_i} + x_iy_i \frac{df}{dy_i} \right\} = \sum_1^n i \left\{ x_iy_i \frac{df}{dx_i} + y_i^2 \frac{df}{dy_i} \right\} = 0,$$

when expressed in terms of I, J, become somewhat complex, viz.:

$$(1)\quad J_4 (I_4 - J_4 - 1) \frac{df}{dJ_4} + \sum_5^n m \left\{ I_m (I_4 J_m - I_m J_4 - J_m + J_4) \frac{df}{dI_m} + \right.$$

$$\left. J_m (I_4 J_m - I_m J_4 - J_m + I_4 - 1) \frac{df}{dJ_m} \right\} = 0.$$

$$(2)\quad I_4 (J_4 - I_4 + 1) \frac{df}{dI_4} + \sum_5^n m \left\{ J_m (I_4 J_m - I_m J_4 - I_m + I_4) \frac{df}{dJ_m} + \right.$$

$$\left. I_m (I_4 J_m - I_m J_4 - I_m + J_4 + 1) \frac{df}{dI_m} \right\} = 0.$$

After considerable manipulation, the solutions of (1) are found to be

$$I_4,\ \varphi_m = \frac{I_m J_4}{J_m},\ \psi_m = \frac{\varphi_m + I_m (I_4 - \varphi_m - 1)}{I_m (J_4 - I_4 + 1)},\quad (m = 5 \ldots . n)\ .$$

With I_4, φ_m, ψ_m as new variables, equation (2) becomes

$$I_4 \frac{df}{dI_4} + \sum_5^n m\ \varphi_m \frac{df}{d\varphi_m} = 0,$$

with solutions

$$Q_m = \frac{\varphi_m}{I_4},\quad \psi_m\ .$$

Selecting as invariants Q_m and $H_m = \frac{1+\psi_m}{Q_m}$, and restoring the variables $x_i\ y_i$, we have

$$Q_m = \frac{|\,1\,2\,m\,|}{|\,1\,2\,4\,|} : \frac{|\,1\,3\,m\,|}{|\,1\,3\,4\,|}, \quad H_m = \frac{|\,1\,2\,4\,|}{|\,1\,2\,m\,|} : \frac{|\,2\,3\,4\,|}{|\,2\,3\,m\,|} .$$

The forms of Q and H show that *the general projective group leaves invariant the cross-ratios of five points.*

Differential Invariants Derived from Point-Invariants.

By David A. Rothrock.

In an accompanying article concerning Point-Invariants, the writer has shown how a group

$$X_k f \equiv \xi_k(x, y)\frac{df}{dx} + \eta_k(x, y)\frac{df}{dy}, \quad (k = 1 \ldots . r),$$

may be *extended* to include the increments of the coördinates of n points. The members of a group may be *extended* in a different manner, and indeed so as to include the increments of

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \ldots .$$

For example, the group $X_k f$ gives to x and y the increments

$$\delta x = \xi_k \delta t, \ \delta y = \eta_k \delta t,$$

and to $y' = \frac{dy}{dx}$, the increment

$$\delta y' = \frac{dx\,.\,\delta dy - dy\,.\,\delta dx}{dx^2} = \frac{d\eta_k - y' d\xi_k}{dx}\delta t \equiv \eta'_k \delta t.$$

Similarly, $y'' = \frac{d^2y}{dx^2}$ receives the increment

$$\delta y'' = \frac{d\eta'_k - y'' d\xi_k}{dx}\delta t \equiv \eta''_k \delta t,$$

and in general

$$\delta y^{(m)} = \frac{d\eta_k^{(m-1)} - y^m d\xi_k}{dx}\delta t \equiv \eta_k^{(m)} \delta t.$$