Selecting as invariants Q_m and $H_m = \frac{1 + \psi_m}{Q_m}$, and restoring the variables $x_i\, y_i$, we have

$$Q_m = \frac{|\,1\,2\,m\,|}{|\,1\,2\,4\,|} : \frac{|\,1\,3\,m\,|}{|\,1\,3\,4\,|}, \quad H_m = \frac{|\,1\,2\,4\,|}{|\,1\,2\,m\,|} : \frac{|\,2\,3\,4\,|}{|\,2\,3\,m\,|} .$$

The forms of Q and H show that *the general projective group leaves invariant the cross-ratios of five points.*

Differential Invariants Derived from Point-Invariants.

By David A. Rothrock.

In an accompanying article concerning Point-Invariants, the writer has shown how a group

$$X_k f \equiv \xi_k(x, y)\,\frac{df}{dx} + \eta_k(x, y)\,\frac{df}{dy}, \quad (k = 1 \,\ldots\, r),$$

may be *extended* to include the increments of the coördinates of n points. The members of a group may be *extended* in a different manner, and indeed so as to include the increments of

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \ldots\ .$$

For example, the group $X_k f$ gives to x and y the increments

$$\delta x = \xi_k \delta t,\ \delta y = \eta_k \delta t,$$

and to $y' = \frac{dy}{dx}$, the increment

$$\delta y' = \frac{dx\,.\,\delta dy - dy\,.\,\delta dx}{dx^2} = \frac{d\eta_k - y' d\xi_k}{dx}\delta t \equiv \eta'_k \delta t.$$

Similarly, $y'' = \frac{d^2y}{dx^2}$ receives the increment

$$\delta y'' = \frac{d\eta'_k - y'' d\xi_k}{dx}\,\delta t \equiv \eta''_k \delta t,$$

and in general

$$\delta y^{(m)} = \frac{d\eta_k{}^{(m-1)} - y^m d\xi_k}{dx}\,\delta t \equiv \eta_k{}^{(m)} \delta t.$$

The group $X_k f$ so extended becomes

$$X_k^{(m)} f = \xi_k \frac{df}{dx} + \eta_k \frac{df}{dy} + \eta_k^{(1)} \frac{df}{dy^1} + \eta_k^{(2)} \frac{df}{dy^{(2)}} + \dots + \eta_k^{(m)} \frac{df}{dy^{(m)}} .*$$

Lie has shown that the extended transformations $X_k^{(m)} f$ form an r — parameter group since the *bracket relations*

$$(X_i^{(m)}, X_k^{(m)}) = X_i^{(m)} (X_k^{(m)} f) — X_k^{(m)} (X_i^{(m)} f) = \sum_1^r {}_s c_{iks} X_s^{(m)} f \dots\dots\dots\dots\dots (1)$$

exist. But when relations (1) hold, the equations

$$X_k^{(m)} f = \xi_k \frac{df}{dx} + \eta_k \frac{df}{dy} + \sum_1^m {}_i \eta_k^{(i)} \frac{df}{dy^{(i)}} = 0$$

are known to form a complete system of linear partial differential equations in $2 + m$ variables. This system has at least $2 + m — r$ independent solutions which are defined as the *differential invariants* of the group $X_k f$.

In Lie's paper cited above it is shown that if two independent differential invariants be known, all others may be found by differentiation. For example, if the two fundamental differential invariants be ϕ_1, ϕ_2, then

$$\phi_3 = \frac{d\phi_2}{d\phi_1}, \ \phi_4 = \frac{d\phi_3}{d\phi_1}, \dots .$$

The fundamental differential invariants $\phi_1 (x, y, y_1, y_2, \dots y_{r-1})$, $\phi_2 (x, y, y_1, y_2, \dots y_r)$, of an r-parameter group may, in general, be obtained from a somewhat different point of view, and indeed without a knowledge of the form of the group itself, provided the point-invariants of the group be known.

Let us suppose the points of a point-invariant $\Theta (x, y, x^{(2)}, y^{(2)} \dots)$ to lie upon a curve $x = f_1 (t)$, $y = f_2 (t)$,
where f_1, f_2 are analytic functions of the parameter t. We seek the nature of the invariants when two or more points upon this curve approach coincidence. If x, y be a point for $t = t_0$, then a point $x^{(2)}$, $y^{(2)}$, ultimately coincident with x, y, will be given by

$$x^{(2)} = x + x' dt + x'' \frac{dt^2}{2} + \dots, \ y^{(2)} = y + y' dt + y'' \frac{dt^2}{2} + \dots, \quad \dagger$$

*Lie: Ueber Differentialgleichungen, die eine Gruppe gestatten. Mathematische Annalen, Bd. XXXII.

†Throughout this paper we shall employ the following notation:

(a) x, y; $x^{(2)}, y^{(2)}$; $x^{(3)}, y^{(3)}$; .... are points of the plane.

(b) $x' = \frac{dx}{dt}$, $x'' = \frac{d^2x}{dt^2}$, ...; $y' = \frac{dy}{dt}$, $y'' = \frac{d^2y}{dt^2}$, ...

(c) $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$, ... ; hence, we have $y' = y_1 x'$,

$y'' = y_2 (x')^2 + y_1 x''$, $y''' = y_3 (x')^3 + 3y_2 x' x'' + y_1 x'''$, ...

and similarly with other parameters for any number of consecutive points. On substituting these series expansions of $x^{(i)}$, $y^{(i)}$ in Θ, we shall evidently obtain an invariant function. If now Θ be capable of expansion in a power-series with regard to dt, dr, ..., we shall have the coefficients, $I_1(x, y, x', y', \ldots)$, $I_2(x_1 y, x', y', \ldots)$, ..., of the powers of dt, dr, ... separately invariant, since the parameters t, r, ... are arbitrary. In I_1, I_2, I_3 .... we may express $y', y'', y''', \ldots$ as functions of $y_1, y_2, \ldots\ x', x'', x''', \ldots\ldots$. If then $I_1, I_2, I_3, \ldots$ may be so combined as to eliminate the differentials $x', x'', x''', \ldots$, we shall obtain invariant functions, $\phi_1(x, y, y_1, y_2, \ldots\ldots)$, ϕ_2, ϕ_3, ..., which are *differential invariants* in the sense already defined.

The calculation of differential invariants by the method just outlined is sometimes quite laborious. Below is given a consideration of some of the more characteristic groups.

SECTION I. DIFFERENTIAL INVARIANTS DETERMINED BY TWO POINTS.

In the present section are computed the differential invariants for some of the more simple groups of the plane, and indeed for such as have point-invariants for two distinct points. Only two differential invariants have been determined for each group; all others may be found from these by differentiation.*

1. The group

$$\boxed{\ q\ }$$

has the point-invariants $x^{(i)}$, $\psi_2 = y - y^{(2)}$. Expressing $y^{(2)}$ in terms of a parameter t, we have ultimately

$$\psi_2 = y - \left(y + y'dt + y''\frac{dt^2}{2} + \ldots.\right).$$

Since dt is arbitrary, $y', y'', \ldots.$ are singly invariant.

$y' = x'y_1$, but x' as well as x is invariant, hence y_1 is invariant, and our differential invariants may be written

$$\phi_1 = x,\ \phi_2 = y_1.$$

2. The group

$$\boxed{\ p, q\ }$$

has the point-invariants

$$u_2 = x - x^{(2)},\ v_2 = y - y^{(2)}.$$

Hence, we have

$$u_2 = x - \left(x + x'dt + x''\frac{dt^2}{2} + \ldots\right),\ v_2 = y - \left(y + y'dt + y''\frac{dt^2}{2} + \ldots\right),$$

*Lie: Math. Annalen, Bd. XXXII, p. 220.

which show $x', x'', \ldots\ldots, y', y'', \ldots\ldots$ to be invariant. But $y' = y_1 x'$, $y'' = y_2(x')^2 + y_1 x''$; hence, y_1, y_2 must each be invariant.

$$\therefore \varphi_1 = y_1, \varphi_2 = y_2 .$$

3. The point-invariants of the group

$$\boxed{q, \ xp + yq}$$

are
$$u_2 = \frac{x^{(2)}}{x}, \ v_2 = \frac{y - y^{(2)}}{x} .$$

Introducing the series expansion of $x^{(2)}, y^{(2)}$,

$$u_2 = (x + x'dt + x''\frac{dt^2}{2} + \ldots) : x,$$

$$v_2 = \left\{ y - (y + y'dt + y''\frac{dt^2}{2} + \ldots) \right\} : x .$$

u_2 shows the ratios

$$\frac{x'}{x}, \frac{x''}{x}, \frac{x'''}{x}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

to be invariant, while v_2 requires the invariance of

$$\frac{y'}{x}, \frac{y''}{x}, \frac{y'''}{x}, \ldots\ldots$$

$$I_1 = \frac{y'}{x} = \frac{y_1 x'}{x};$$

hence y_1 is invariant on account of (1).

$$I_2 = \frac{y''}{x} = \frac{y_2 (x')^2 + y_1 x''}{x}, \text{ or } I_2 - \varphi_1 \frac{x''}{x} = xy_2 \left(\frac{x'}{x}\right)^2 .$$

Therefore, $\varphi_1 = y_1, \varphi_2 = xy_2$.

4. The group

$$\boxed{p, \ q, \ xp + yq}$$

has the point-invariants

$$u_2 = \frac{y - y^{(2)}}{x - x^{(2)}}, \ v_3 = \frac{x - x^{(3)}}{x - x^{(2)}} .$$

One differential invariant may be computed from u_2 alone, but a second can not be had on account of impossibility of the elimination of the parameters. We therefore consider three points determined by t, r.

$$u_2 = \left\{ y - (y + y' dt + y''\frac{dt^2}{2} + \ldots) \right\} : \left\{ x - (x + x' dt + x''\frac{dt^2}{2} + \ldots) \right\}$$

$$= \frac{y'}{x'} + \frac{dt}{2}\left(\frac{y''}{x'} - \frac{y' x''}{(x')^2}\right) + \frac{dt^2}{2}\left(\frac{y' (x'')^2}{2 (x')^3} - \frac{y' x'''}{3 (x')^2} - \frac{y'' x''}{2 (x')^2} + \frac{y''}{3x'}\right) + \ldots,$$

$$v_3 = \left\{x - (x + x'\,dr + x''\frac{dr^2}{2} + \ldots.)\right\} : \left\{x - x\,(+ x'\,dt + x''\frac{dt^2}{2} + \ldots.\right\}$$

$$= \frac{dr}{dt} - \frac{dr}{2}\cdot\frac{x''}{x'} - \frac{dr^2}{4}\left(\frac{x''}{x'}\right)^2 + dt\,dr\left\{\left(\frac{x''}{2x'}\right)^2 - \frac{x'''}{6x'}\right\} + \ldots\ldots.$$

These functions show

$$\frac{x''}{x'},\ I_1 = \frac{y'}{x'} = y_1,\ I_2 = \frac{y''}{x'} - \frac{y'x''}{(x')^2} = y_2\,x',\ \text{and}$$

$$I_3 = \frac{y'''}{3\,x'} - \frac{y''\,x''}{2\,(x')^2} - \frac{y'\,x'''}{3\,(x')^2} + \frac{y'\,(x'')^2}{2\,(x')^3} = \frac{y_3\,(x')^3}{3} + \frac{y_2 x''}{2}$$

to be invariant. Eliminating the parameters x', x'', we have

$$\left\{I_3 \div I_2 - \frac{x''}{2x'}\right\} \div I_2 = \frac{y_3}{3\,(y_2)^2}.$$

$$\therefore \phi_1 = y_1,\ \phi_2 = \frac{y_3}{y_2{}^2}.$$

SECTION II. DIFFERENTIAL INVARIANTS DETERMINED BY THREE OR MORE POINTS.

In the case of the more complex groups it is necessary to bring into consideration three, four, five, ... points, and consequently employ additional parameters, r, s, ...

5. For three points, the group

$$\boxed{p,\ q,\ xp + cyq}$$

possesses the point-invariants

$$u = \frac{y - y^{(2)}}{(x - x^{(2)})^c},\quad v_3 = \frac{x - x^{(3)}}{x - x^{(2)}},\quad w_3 = \frac{y - y^{(3)}}{y - y^{(2)}}\ .$$

Expressing u, in series expansion for $x^{(2)}$, $y^{(2)}$, we have

$$u = \frac{y - (y + y'dt + y''\frac{dt^2}{2} + .)}{\left\{x - (x + x'dt + x''\frac{dt^2}{2} + ..)\right\}^c}$$

$$= \frac{k}{(x')^c}\left\{y' + \frac{dt}{2}\left[y'' - cy'\frac{x''}{x'}\right] + \right.$$

$$\left. dt^2\left[\frac{y'''}{6} - \frac{cy''}{4}\cdot\frac{x''}{x'} + y'\left(l\left(\frac{x''}{x'}\right)^2 - \frac{c}{6}\,\frac{x'''}{x'}\right)\right]\ldots.\right\}.$$

The series expansion of v_3 is identical with that of v_3 in 4 above. Hence, the invariant functions may be written

$$\frac{x''}{x'},\ \frac{x'''}{x'},\ \frac{x^{iv}}{x'},\ \ldots\ldots,\ I_1 = \frac{y'}{(x')^c} = \frac{v_1}{(x')^{c-1}},\ I_2 = \frac{y''}{(x')^c} - cy'\frac{x''}{(x')^{c+1}} =$$

$$\frac{v_2}{(x')^{c-2}} - h\,.\,I_1\,\frac{x''}{x'},$$

$$I_3 = \frac{y'''}{6\,(x')^c} - \frac{cy''}{4\,(x')^c}\cdot\frac{x''}{x'} + \frac{y'}{(x')^c}\left\{ l \left(\frac{x''}{x'} \right)^2 - \frac{c}{6}\cdot\frac{x'''}{x'} \right\}$$

$$= k_1\frac{v_3}{(x')^{c-3}} + k_2\frac{x''}{x'}\cdot\frac{v_2}{(x')^{c-2}} + \left\{ k_3\,\frac{x'''}{x'} + k_4\left(\frac{x''}{x'} \right)^2 \right\}\frac{v_1}{(x')^{c-1}}$$

From these relations follows at once the invariance of

$$\frac{v_1}{(x')^{c-1}},\ \frac{v_2}{(x')^{c-2}},\ \frac{v_3}{(x')^{c-3}}.$$

By eliminating x', we have

$$\omega_1 = \frac{v_2}{v_1^{\frac{c-2}{c-1}}},\quad \omega_2 = \frac{v_3}{v_1^{\frac{c-3}{c-1}}}.$$

6. $\overline{q,\ yq}$ leaves invariant x and $v_3 = \frac{y - y^{(3)}}{y - y^{(2)}}$. Expanding v_3 in series,

$$v_3 = \left\{ y - \left(y + y'\,dr + y''\frac{dr^2}{2} \ldots\ldots\right) \right\} : \left\{ y - \left(y + y'\,dt + y''\frac{dt^2}{2} + \ldots\ldots\right) \right\}$$

$$= \frac{dr}{dt} - \frac{dr}{2}\,\frac{y''}{y'} - dr^2\left(\frac{y''}{2y'} \right)^2 + \ldots\ldots,$$

which gives invariant functions $\frac{y''}{y'}, \frac{y'''}{y'}, \ldots\ldots$ The functions $x,\ x',\ x''\ \ldots\ldots$ are also invariant.

$$I_1 = \frac{y''}{y'} = \frac{v_2}{v_1}\,x' + \frac{x''}{x'}.$$

$$\therefore\ \omega_1 = x,\ \omega_2 = \frac{v_2}{v_1}.$$

7. The group

$$\overline{q,\ yq,\ p}$$

has point-invariants

$$u_2 = x - x^{(2)},\ v_3 = \frac{y - y^{(3)}}{y - y^{(2)}}\ .$$

We have, as in 6, the invariant functions

$$x',\ x'',\ x''',\ \ldots,\ I_1 = \frac{y''}{y'} = \frac{y_2 x'}{y_1} + \frac{x''}{x'},$$

$$I_2 = \frac{y'''}{y'} = \frac{y_3 (x')^2}{y_1} + 3\,\frac{y_2 x''}{y'} + \frac{x'''}{x'}.$$

$$\therefore \varphi_1 = \frac{y_2}{y_1},\quad \varphi_2 = \frac{y_3}{y_1}.$$

8. The point-invariants of the four-parameter group

$$\boxed{p,\ xp,\ q,\ yq}$$

are
$$u_3 = \frac{x - x^{(3)}}{x - x^{(2)}},\quad v_3 = \frac{y - y^{(3)}}{y - y^{(2)}}.$$

The series expansion for u_3, v_3 in powers of dt, dr will be identical with those for v_3 in 4 and 7, respectively. Hence, we have the invariant differential functions

$$\frac{x''}{x'},\ \frac{x'''}{x'},\ \frac{x^{iv}}{x'},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \quad (1),$$

and

$$I_1 = \frac{y''}{y'} = \frac{y_2 x'}{y_1} + \frac{x''}{x'},\quad I_2 = \frac{y'''}{y'} = \frac{y_3 (x')^2}{y_1} + 3\frac{y_2 x'}{y_1}\cdot\frac{x''}{x'} + \frac{x'''}{x'},$$

$$I_3 = \frac{y^{iv}}{y'} = \frac{y_4 (x')^3}{y_1} + \frac{6\, y_3 (x')^2}{y_1}\cdot\frac{x''}{x'} + \frac{y_2 x'}{y_1}\left\{3\left(\frac{x''}{x'}\right)^2 + 4\,\frac{x'''}{x'}\right\} + \frac{x^{iv}}{x'}.$$

Hence, on account of (1), we have the invariant functions

$$\frac{y_2 x'}{y_1},\ \frac{y_3 (x')^2}{y_1},\ \frac{y_4 (x')^3}{y_1},$$

from which it is only necessary to eliminate x' in order to obtain our required differential invariants:

$$\varphi_1 = \frac{y_1 y_3}{y_2^{\,2}},\quad \varphi_2 = \frac{y_4 y_1^{\,2}}{y_2^{\,3}}.$$

9. The general projective group in one variable

$$\boxed{q,\ yq,\ y^2 q}$$

leaves invariant x and $R = \dfrac{y^{(2)} - y^{(4)}}{y - y^{(4)}} : \dfrac{y^{(2)} - y^{(3)}}{y - y^{(3)}}$.

Using t, r, s as auxiliary variables, R takes the form, for ultimately coincident points

$$R = \frac{1 - \alpha}{1 - \beta} = (1 - \alpha)\,(1 + \beta + \beta^2 + \ldots),$$

where $\alpha = (y' \, dt + y'' \dfrac{dt^2}{2} + \ldots.) : (y' ds + y'' \dfrac{ds^2}{2} + \ldots.)$, and

$$\beta = (y' \, dt + y'' \frac{dt^2}{2} + \ldots.) : (y' \, dr + y'' \frac{dr^2}{2} + \ldots.).$$

Arranging R according to positive powers of dt, dr, ds, and omitting superfluous terms, we find

$$R \equiv \ldots. \, dt \, (ds - dr) \left\{ \frac{y'''}{6y'} - \left(\frac{y''}{2y'} \right)^2 \right\} + \ldots.$$

$$+ \, dt \, (ds^2 - dr^2) \left\{ \frac{y^{iv}}{24y'} - \frac{y'' \, y'''}{6 \, (y')^2} + \left(\frac{y''}{2y'} \right)^3 \right\} + \ldots.$$

$$+ \, dt \, (ds^3 - dr^3) \left\{ \frac{y^{v}}{120y'} - \frac{y'' \, y^{iv}}{24 \, (y')^2} - \left(\frac{y'''}{6y'} \right)^2 - \left(\frac{y''}{2y'} \right)^4 + \frac{(y'')^2 \, y'''}{8 \, (y')^3} \right\} + \ldots$$

From these coefficients we may determine the differential invariants.

$\phi_1 = x.$

$$I_1 = \frac{y'''}{6y'} - \left(\frac{y''}{2y'} \right)^2 = \frac{(x')^2}{12} \, \frac{2y_1 y_3 - 3y_2^2}{y_1^2} + \frac{x'''}{6x'} - \left(\frac{x''}{2x'} \right)^2,$$

$$\therefore \varphi_2 = \frac{2y_1 y_3 - 3y_2^2}{y_1^2}.$$

$$I_2 = \frac{(x')^3}{24} \left\{ \frac{y_4}{y_1} - \frac{4y_2 \, y_3}{y_1^2} + \frac{3y_2^3}{y_1^3} \right\} + \frac{x' \, x''}{24} \varphi_2 + I_2 \, (x),$$

$$\therefore \varphi_3 = \frac{y_4}{y_1} - 4 \, \frac{y_2 \, y_3}{y_1^2} + 3 \, \frac{y_2^3}{y_1^3}.$$

$$I_3 = \frac{(x')^4}{120} \left\{ \frac{y_5}{y_1} - 5 \, \frac{y_2 \, y_4}{y_1^2} - 4 \, \frac{y_3^2}{y_1^2} + 17 \, \frac{y_2^2 \, y_3}{y_1^3} - 9 \, \frac{y_2^4}{y_1^4} \right\} +$$

$$+ \frac{(x')^2 \, x''}{24} \cdot \varphi_3 + \frac{(x')^4}{720} \varphi_2^2 - \frac{x' \, x'''}{72} \varphi_2 + I_3 \, (x),$$

$$\therefore \phi_4 = \frac{y_5}{y_1} - 5 \, \frac{y_2 \, y_4}{y_1^2} - 4 \left(\frac{y_3}{y_1} \right)^2 + 17 \, \frac{y_2^2 \, y_3}{y_1^3} - 9 \left(\frac{y_2}{y_1} \right)^4.$$

In some of the following paragraphs we shall need the forms I_2, I_3, here computed. Incidentally we have computed the differential invariants φ_3, φ_4.

10. The group

$$\boxed{q, \; yq, \; y^2 q, \; p}$$

has the same differential invariants as 9 above, with the exception of ϕ_1, which must be omitted. We shall have, therefore, ϕ_2, φ_3, φ_4, as defined above.

11. By the group

$$\boxed{q,\ yq,\ y^2q,\ p,\ xp}$$

the functions I_1, I_2, I_3 of 9 remain invariant, also $\frac{x''}{x'}, \frac{x'''}{x'}, \frac{x^{iv}}{x'}, \ldots$ as in 8. These invariant functions must be so manipulated that the x's are either eliminated or made to appear as ratios $\frac{x''}{x'}, \frac{x'''}{x'}, \ldots$ Since $I_1(x)$, $I_2(x)$ .... are already functions of $\frac{x''}{x'}, \frac{x'''}{x'}, \ldots$, we may omit these, and write simply

$$J_1 = \phi_2(x')^2,\ J_2 = \varphi_3(x')^3 + \phi_2 x'x'',$$

$$J_3 = \varphi_4 \frac{(x')^4}{5} + \phi_3(x')^2x'' + \varphi_2{}^2\frac{(x')^4}{30} + \phi_2\frac{x'x'''}{3}.$$

Eliminating x', x'', ... ,

$$\left\{J_2 : J_1 - \frac{x''}{x'}\right\} : (J_1)^{\frac{1}{2}} \equiv \frac{\varphi_3}{(\varphi_2)^{\frac{3}{2}}} - \frac{\frac{y_4}{y_1} - \frac{4y_2y_3}{y_1{}^2} + 3\left[\frac{y_2}{y_1}\right]^3}{\left\{\frac{2y_3}{y_1} - 3\left[\frac{y_2}{y_1}\right]^2\right\}^{\frac{3}{2}}} = \Phi_1 .$$

$$J_3 : J_1 \equiv \frac{\varphi_4}{\varphi_2}\cdot\frac{(x')^2}{5} + \frac{\varphi_3}{\varphi_2}x'\left\{\frac{x''}{x'}\right\} + \phi_2\frac{(x')^2}{30} + \frac{x'''}{3x'}$$

$$= \frac{\phi_4}{\varphi_2}\cdot\frac{(x')^2}{5} + \left\{J_2 : J_1 - \frac{x''}{x'}\right\}\frac{x''}{x'} + \frac{J_1}{30} + \frac{x'''}{3x'}.$$

Hence, $A \equiv \frac{\varphi_4}{\phi_2}(x')^2$ is invariant.

$$A : J_1 = \frac{\phi_4}{\phi_2{}^2} = \frac{\frac{y_5}{y_1} - 5\frac{y_2y_4}{y_1{}^2} - 4\left[\frac{y_3}{y_1}\right]^2 + 17\frac{y_2{}^2y_3}{y_1{}^3} - 9\left[\frac{y_2}{y_1}\right]^4}{\left\{\frac{2y_3}{y_1} - 3\left[\frac{y_2}{y_1}\right]^2\right\}^2} = \Phi_2 .$$

Φ_1, Φ_2 are the two fundamental differential invariants.

12. It has been shown that the group

$$\boxed{\begin{gathered} X_1(x)\,q,\ X_1(x)\,.\,q,\ X_3(x)\,q,\ \ldots\ X_r(x)\,.\,q \\ r > 1 \end{gathered}}$$

leaves invariant x and the determinant

$$D = \begin{vmatrix} y & y^{(2)} & y^{(3)} & \ldots\ldots & y^{(r+1)} \\ X_1(x) & X_1(x^{(2)}) & X_1(x^{(3)}) & \ldots\ldots & X_1(x^{(r+1)}) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ X_r(x) & X_r(x^{(2)}) & X_r(x^{(3)}) & \ldots\ldots & X_r(x^{(r+1)}) \end{vmatrix}.$$

We shall denote the parameters for $x^{(2)}$, $x^{(3)}$ .... by t, s, ..., respectively, and have series expansion for $X_i(x^{(2)})$ in the form

$$X_i(x^{(2)}) = X_i(x + x'dt + x''\frac{dt^2}{2} + x'''\frac{dt^3}{6} + ..)$$

$$= X_i(x) + X_i'(x).x'dt + \left\{X_i''(x).x'^2 + X_i'(x).x''\right\}\frac{dt^2}{2} +$$

$$+ \left\{X_i'''(x).x'^3 + 3X_i''(x).x'x'' + X_i'(x).x'''\right\}\frac{dt^3}{6} +$$

$$+ \left\{X_i^{iv}(x).x'^4 + 6X_i'''(x).x'^2x'' + 3X_i''(x).x''^2 +\right.$$

$$\left. + 4X_i''(x).x'x''' + X_i'(x).x^{iv}\right\}\frac{dt^4}{24} + \ldots\ldots\ldots\ldots$$

with like expansions for $X_i(x^{(3)})$, ... in parameters s, ... Substituting these series expansions for X_i in the above determinant and subtracting vertical columns in a proper manner, we have

$$\begin{vmatrix} y & y_1x' + \ldots & y_2(x')^2 + \ldots & y_3(x')^3 + \ldots & \ldots\, y_{r+1}(x')^{r+1} + \ldots \\ X_1 & X_1x' + \ldots & X_1''.(x')^2 + \ldots & X_1'''.(x')^3 + \ldots & \ldots\, X_1^{r+1}.(x')^{r+1} + \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ X_r & X_r'x' & \ldots & \ldots & \ldots \end{vmatrix}$$

Or disregarding x', x'', ... which are invariant, and retaining only the elements of lowest degree in dt, ds, ..., we have

$$\phi_1 = \begin{vmatrix} y & y_1 & y_2, & \ldots & y_{r+1} \\ X_1 & X_1' & X_1'', & \ldots & X_1^{r+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ X_r & X_r' & X_r'' & \ldots & X_r^{r+1} \end{vmatrix}.$$

Since x is also invariant, $\phi^2 = \frac{d\phi_1}{dx}$, which would be the above determinant with the last column changed to y_{r+2}, X_1^{r+2}, ... X_r^{r+2}.

13. $\boxed{X_1 q,\ X_2 q,\ \ldots\ldots\ X_{r-1} q,\ yq}$ leaves invariant x and the ratio $\phi_2 : \phi_1$, where ϕ_2, ϕ_1 are determinants defined in 12.

Since x also remains invariant, we may write our differential invariants

$$\Phi_1 = \frac{\phi_2}{\phi_1}.$$

$$\Phi_2 = \frac{d\Phi_1}{dx}.$$

14. The special linear group

$$\boxed{p,\ q,\ xq,\ xp - yq,\ yp}$$

has the point-invariant

$$D = \begin{vmatrix} x & y & 1 \\ x^{(2)} & y^{(2)} & 1 \\ x^{(3)} & y^{(3)} & 1 \end{vmatrix}.$$

Expressing $x^{(2)}, y^{(2)}$; $x^{(3)}, y^{(3)}$ in series expansion in terms of t, s,

$$D = \begin{vmatrix} x & y & 1 \\ x + x'\,dt + x''\frac{dt^2}{2} + \ldots, & y + y'\,dt + y''\frac{dt^2}{2} + \ldots, & 1 \\ x + x'\,ds + x''\frac{ds^2}{2} + \ldots, & y + y'\,ds + y''\frac{ds^2}{2} + \ldots, & 1 \end{vmatrix}$$

$$= I_1\frac{dt\,ds^2}{2} + I_2\frac{dt\,ds^3}{6} + I_3\frac{dt\,ds^4}{24} - I_4\frac{dt^2\,ds^2}{12} + I_5\frac{dt\,ds^5}{120} -$$

$$- I_6\frac{dt^2\,ds^4}{48} + \ldots,$$

where

$I_1 = x'\,y'' - x''\,y' = y_2\,(x')^3,$

$I_2 = x'\,y''' - x'''\,y' = y_3\,(x')^4 + 3y_2\,(x')^2\,x'',$

$I_3 = x'\,y^{iv} - x^{iv}\,y' = y_4\,(x')^5 + 6y_3\,(x')^3 x'' + 3y_2\,x'\,(x'')^2 + 4y_2\,(x')^2\,x''',$

$I_4 = x'''\,y'' - x''\,y''' = y_2\,[\,(x')^2\,x''' - 3x'\,(x'')^2] - y_3\,(x')^3\,x'',$

$I_5 = x'\,y^{v} - x^{v}\,y' = y_5\,(x')^6 + 10y_4\,(x')^4\,x'' + 15\,(x'\,x'')^2 + 10y_3\,(x')^3\,x''' +$
$\quad + 10y_2\,x'\,x''\,x''' + 5y_2\,(x')^2\,x^{iv},$

$I_6 = x''\,y^{iv} - x^{iv}\,y'' = y_4\,(x')^4\,x'' + 6y_3\,(x'\,x'')^2 + y_2\,[3\,(x'')^3 + 4x'\,x''\,x''' -$
$\quad - (x')^2\,x^{iv}\,].$

From these six invariant functions we eliminate the differentials x' x'', ... obtaining the differential invariants:

$$\Phi_1 = \left\{3\, I_1 I_3 - 12\, I_1 I_4 - 5\, I_2^{\,2}\right\} : (I_1)^{\frac{8}{3}} = (3y_2 y_4 - 5y_3^{\,2}) : y_2^{\frac{8}{3}}.$$

$$\Phi_2 = \left\{15\, I_1^{\,2} I_6 + 3\, I_1^{\,2} I_5 + \tfrac{40}{3} I_2^{\,3} - 15\, I_1 I_2 (I_3 - 2\, I_4)\right\} : I_1^{\,4}$$

$$= \left\{3y_2^{\,2} y_5 - 15 y_2 y_3 y_4 + \tfrac{40}{3} y_3^{\,3}\right\} : y_2^{\,4}.$$

15. The general linear group

$$\boxed{p,\ q,\ xq,\ xp - yq,\ yp,\ xp + yq}$$

leaves invariant the quotient

$$Q = \begin{vmatrix} x & y & 1 \\ x^{(2)} & y^{(2)} & 1 \\ x^{(3)} & y^{(3)} & 1 \end{vmatrix} : \begin{vmatrix} x & y & 1 \\ x^{(3)} & y^{(3)} & 1 \\ x^{(4)} & y^{(4)} & 1 \end{vmatrix}.$$

Using t, s, r as parameters of three successive points, we find

$$Q = \begin{vmatrix} x & y & 1 \\ x + x'dt + \dots & y + y'dt + \dots & 1 \\ x + x'ds + \dots & y + y'ds + \dots & 1 \end{vmatrix} : \begin{vmatrix} x & y & 1 \\ x + x'ds + \dots & y + y'ds + \dots & 1 \\ x + x'dr + \dots & y + y'dr + \dots & 1 \end{vmatrix}$$

$$= \left\{ \begin{array}{l} k_1 I_1 \,|\,dt\, ds^2\,| + k_2 I_2 \,|\,dt\, ds^3\,| + k_3 I_3 \,|\,dt\, ds^4\,| + k_4 I_4 \,|\,dt^3 ds^2\,| + \\ + k_5 I_5 \,|\,dt\, ds^5\,| + k_6 I_6 \,|\,dt^2 ds^4\,| + k_7 I_7 \,|\,dt\, ds^6\,| + k_8 I_8 \,|\,dt^2\, ds^5\,| + \\ \qquad + \ k_9 I_9 \,|\,dt^3\, ds^4\,| \end{array} \right\} :$$

$$: \left\{ \text{Similar expression in } ds,\ dr. \right\},$$

where k_i are constants, $|\,dt^a\, ds^b\,| = \begin{vmatrix} dt^a & dt^b \\ ds^a & ds^b \end{vmatrix}$, and I_i are functions defined as in 14. The form of this expansion for Q shows at once the invariance of the quotients $I_2 : I_1$, $I_3 : I_1$, ... Denoting these ratios by R, we have

$$R_2 = I_2 : I_1 = (x'y''' - x'''y') : (x'y'' - x''y'),$$
$$R_3 = I_3 : I_1 = (x'y^{iv} - x^{iv}y') : I_1,$$
$$R_4 = I_4 : I_1 = (x'''y'' - x''y''') : I_1,$$
$$R_5 = I_5 : I_1 = (x'y^{v} - x^{v}y') : I_1,$$
$$R_6 = I_6 : I_1 = (x''y^{iv} - x^{iv}y'') : I_1,$$
$$R_7 = I_7 : I_1 = (x'y^{vi} - x^{vi}y') : I_1,$$
$$R_8 = I_8 : I_1 = (x''y^{v} - x^{v}y'') : I_1,$$
$$R_9 = I_9 : I_1 = (x'''y^{iv} - x^{iv}y''') : I_1.$$

In these eight functions we must express y^i in terms of y_i and x^i, and then eliminate the differentials x', x'', ... This work of elimination is quite tedious, but may be briefly indicated. We construct three functions.

$$A \equiv 3\,R_3 - 12\,R_4 - 5R_2^{\,2} = \frac{3y_2y_4 - 5y_3^{\,2}}{y_2^{\,2}}\,(x')^2,$$

$$B \equiv 15\,R_6 + 3R_5 + \frac{40}{3}\,R_2^{\,3} - 15\,R_2\,R_3 + 30\,R_2\,R_4$$

$$\equiv \frac{3y_2^{\,2}y_5 - 15y_2y_3y_4 + \frac{40}{3}\,y_3^{\,3}}{y_2^{\,3}}\,(x')^3,$$

$$C \equiv 18R_8 + 3R_4 - 60\,R_9 - 21\,R_2\,R_5 - \tfrac{35}{3}\,R_2^{\,4} + 35\,R_2^{\,2}R_3 + 70\,R_2^{\,2}R_4 + 210R_4^{\,2}$$

$$\equiv \frac{3y_2^{\,3}y_6 - 21y_2^{\,2}y_3y_5 + 35\,y_2\,y_3^{\,2}y_4 - \frac{35}{3}\,y_3^{\,4}}{y_2^{\,4}}\,(x')^4,$$

and eliminate from these x', giving the differential invariants

$$\Phi_1 = (3y_2^{\,2}y_5 - 15\,y_2y_3y_4 + \tfrac{40}{3}\,y_3^{\,3}) : (3y_2y_4 - 5y_3^{\,2})^{\frac{3}{2}}$$

$$\Phi_2 = (3y_2^{\,3}y_6 - 21\,y_2^{\,2}y_3y_5 + 35y_2y_3^{\,2}y_4 - \tfrac{35}{3}\,y_3^{\,4}) : (3y_2y_4 - 5y_3^{\,2})^2.$$

Mathematical Definitions. By Moses C. Stevens.

Performance of the Twenty-Million-Gallon Snow Pumping Engine of the Indianapolis Water Company. By W. F. M. Goss.

The fact that a pumping engine recently installed within the State of Indiana has given a duty performance higher than that previously reported for any pumping engine in any country is deemed of sufficient moment to merit the attention of the Academy.

This engine was built by the Snow Steam Pump Works of Buffalo, N. Y., and its installation at the Riverside station of the Indianapolis Water Company was completed in season for an acceptance test in July, 1898. It is a triple-expansion, fly-wheel engine, having a single acting pump below and in line with each of the three steam cylinders. Its principal dimensions are as follows:

Diameter of cylinders:	*Inches.*
High pressure	29
Intermediate	52
Low pressure	80