A Linear Relation Between Certain of Klein's X Functions and Sigma Functions of Lower Division Value. By John A. Miller.

Professor Felix Klein has defined a system of interesting functions by the following equation*:

$$X_{\frac{a}{m}}(u) = C_a \prod_{\mu=0}^{m-1} \sigma_{\frac{a}{m}+e,\ \frac{\mu+e}{m}} (u \mid \omega_1, \omega_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where $\varepsilon = 0$, or $\frac{1}{2}$, according as m is odd or even, and

$$\sigma_{\frac{\lambda}{m}, \frac{\mu}{m}}(u \mid \omega_1, \omega_2) = e^{\left\{\frac{\lambda\eta_1 + \mu\eta_2}{m}\right\}\left\{\frac{u - \lambda\omega_1 - \mu\omega_2}{2m}\right\}} \sigma\left(u - \frac{\lambda\omega_1 + \mu\omega_2}{m} \mid \omega_1, \omega_2\right) \ldots (2)$$

u is the fundamental variable of the elliptic functions, ω_1, ω_2 the periods of an elliptic integral of the first kind, η_1, η_2 the periods of an elliptic integral of the second kind, C_a, a quantity independent of u and $\sigma(u \mid \omega_1, \omega_2)$ is the ordinary Weierstrassian σ-function and where λ, μ and m are integers.

I shall now prove that in the case m is a square number, *i. e.*, $m = n^2$ that

$X_{\frac{a}{n^2}}(u)$ *can be expressed as a linear homogeneous function of* $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu \mid \omega_1, \omega_2)$.

To do this, we need the so-called *Hermite Law*† which, when specialized to meet our needs, is as follows:

Suppose we are given n quantities defined as follows:

$$x_i = C_i \prod_{j=1}^{n} \sigma(u - a_{i,j}) \qquad (i = 1 \ldots n)$$

such that the sum of the zero points in the period parallelogram of the u-plane is,

$$S = \Sigma a_{i,j} = 0,$$

And, suppose that we are given a σ-product,

$$P = f(\omega) e^{(\lambda n_1 + \mu n_2)\left(u + \frac{\lambda\omega_1 + \mu\omega_2}{2}\right)} \prod_{i=1}^{n} (u - u_i)$$

Such that the sum of the vanishing points of P in the period parallelogram of the u-plane is

$$\sum_{i=1}^{n} u_i = \lambda\omega_1 + \mu\omega_2$$

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* See Felix Klein: "Vorlesungen über die Theorie der Elliptischen Modulfunctionen," Zweiter Band, p. 261, equation 1.

†F. Klein, Elliptische Normal-Curven und Modeln nter Stufe, p. 355, or Crelle's Journal, Band XXXII, Hermites, Lettre à Mr. Jacobi.

Then P can be expressed as a linear homogeneous function of x_i.

Proof:

$$*\sigma(nu \mid \omega_1 ; \omega_2) = f(\omega_1, \omega_2) \Pi\, \sigma\left(u - \frac{m_1 \omega_1 + m_2 \omega_2}{n} \mid \omega_1, \omega_2\right) \cdot$$

$$e^{\left(\frac{\eta_1 + \eta_2}{2}\right) n(n-1)u} \qquad \ldots\ldots(3)$$

$$m_1, m_2 = 0 \ldots n-1$$

$$\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu, \mid \omega_1, \omega_2) = e^{\left(\frac{\lambda n_1 + \mu n_2}{n}\right)\left(nu - \frac{\lambda\omega_1 + \mu\omega_2}{2n}\right)} \sigma\left(nu - \frac{\lambda\omega_1 + \mu\omega_2}{n} \mid \omega_1, \omega_2\right)$$

[From (2)

$$\therefore \sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu \mid \omega_1, \omega_2) =$$

$$e^{\frac{(\lambda n_1 + \mu n_2)}{n}\left(nu - \frac{\lambda\omega_1 + \mu\omega_2}{2n}\right)}\, e^{\frac{\eta_1 + \eta_2}{2} \cdot n \cdot (n-1)\left(u - \frac{\lambda\omega_1 + \mu\omega_2}{\eta^2}\right)}$$

$$\cdot f_1(\omega), \omega_2) \prod_{\left.\begin{matrix} m_1 \\ m_9 \end{matrix}\right\} = 0}^{n-1} \sigma\left(u - \frac{\lambda\omega_1 + \mu\omega_2}{n^2} - \frac{m_1\omega_1 + m_2\omega_2}{n} \mid \omega_1, \omega_2\right)$$

from equations (2) and (3).

Whence $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu \mid \omega_1, \omega_2)$ is a σ-product of n^2 factors.

Whose residue sum,

$$S = \lambda\omega_1 + \mu\omega_2 + n\frac{(n-1)}{2}\omega_1 + n\frac{(n-1)}{2}\omega_2\,;$$

moreover there are n^2 different quantities

$$\dagger\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu \mid \omega_1, \omega_2) \cdot$$

If now, m define n^2 quantities

$$xi = C_1, \prod_{j=0}^{n^2-1} \sigma(u - u_2 j) \qquad \ldots\ldots(4)$$

Such that $\sum_{j=0}^{n_2-1} a_{i,j} = 0 \quad (i = 0 \ldots . n^2 - 1)$

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*Jordan: "Cours d'Analyse," Tome 2, p. 388.

† Klein: "Vorlesungen über die Theorie der Elliptischen Modulfunctionen," Vol. II, p. 26.

then by Hermites Law, each of the n^2 $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu \mid \omega_1, \omega_2)$

can be expressed as a linear homogeneous function of xi.

We must now divide our discussion into two cases (a)

$n = 1$ (mod. 2)

$$X_{\frac{a}{n^2}}(u) = f_1(\omega_1, \omega_2) \prod_{\mu = 0}^{n^2-1} \sigma_{\frac{a}{n^2}, \frac{\mu}{n^2}}(u \mid \omega_1, \omega_2)$$

$$= f_3(\omega_1, \omega_2)\, e^{a\eta_1 + \left(\frac{n^2-1}{2}\right)\eta_2 \left\{u - \left(a\omega_1 + \frac{\eta^2-1}{2}\omega_2\right)\right\}}$$

$$\cdot \prod_{\mu = 0}^{n^2-1} \sigma\left(u - \frac{a\omega_1 + \mu\omega_2}{n^2} \mid \omega_1, \omega_2\right),$$

Whence $X_{\frac{a}{n^2}}(u)$ is a σ-product of n^2 factors whose residual sum

$$S = a\omega_1 + \frac{n^2-1}{2}\omega_2.$$

And hence can be expressed as a linear function of x_i defined in equation (4).

*There are n^2 quantities $X_{\frac{a}{n^2}}(u)$.

We have now shown that we can express the n^2 quantities $X_{\frac{a}{n^2}}(u)$ as linear homogeneous functions of x_i, and also the n^2 quantities $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu \mid \omega_1, \omega_2)$ as linear homogeneous functions of x_i, whence we can express $X_{\frac{a}{n^2}}(u)$ as a linear homogeneous function of $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu \mid \omega_1, \omega_2)$.

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(b) $n = 0$ (mod. 2).

In this case,

$$X_{\frac{a}{n^2}}(u) = f(\omega_1, \omega_2) \prod_{\mu = 0}^{n^2-1} \sigma_{\frac{a}{n^2} + \frac{1}{2}, \frac{\mu - \frac{1}{2}}{n^2}}(u \mid \omega_1\omega_2) \quad \text{(Equation (1)}$$

$$= f_1(\omega_1, \omega_2) e^{\left\{n^2\left(\frac{a}{n^2} + \frac{1}{2}\right)\eta_1 + \frac{n^2}{2}\eta_2\right\}\left\{\frac{u - n^2\left(\frac{a}{n^2} + \frac{1}{2}\right)\omega_1 + \frac{n^2}{2}\omega_2}{2}\right.}$$

*Klein: Vorlesungen, etc., Vol. II, p. 264.

$$\cdot \prod_{\mu=0}^{n^2-} \sigma\left(u - \frac{a\,\omega_1 + \mu\,\omega_2}{n_2} \mid \omega_1, \omega_2\right)$$

Whence $X_{\frac{a}{n^2}}(u)$ is a σ-product of n^2 factors whose residue sum,

$S \quad a\,\omega_1 + \frac{n^2-1}{2}\,\omega_2$ and hence can be expressed as a linear homogeneous function of x i defined in equation (4)

By repetition of the argument made in case (a) it follows that $X_{\frac{a}{n^2}}(u)$ can be expressed as a linear homogeneous function of $\sigma_{\frac{\lambda}{u}, \frac{\mu}{n}}(nu \mid \omega_1\ \omega_2)$.

Hence our proposition is proved for all integral values of n.

A Formula for the Deflection of Car Bolsters.* By W. K. Hatt.

The body bolster of a car is a beam which carries the weight of the car and its loading and transfers this weight to the center of the truck bolster, which, in turn, transfers the weight to the wheels.

The bolsters are either of trussed form or of beam form. In the latter case they are of I section or else with one flange and web plates.

It is quite important to construct the body bolster so that it may be stiff enough to prevent contact at the side bearings. These side bearings are placed between the truck and body bolster to limit the oscillations of the car. Evidently if the side bearings come into contact the consequent friction will offer additional resistance when the car goes around curves.

The problem is to compute the deflection of a beam of variable depth.

In case of beam bolsters the moment of inertia of the cross section may be taken to be a linear function of the distance of the cross section from the free end of the beam.

Referring to Fig. 1, let AB be one-half of a body bolster and OB the curve into which the half-bolster is bent. Any point of this curve is located with reference to O by its co-ordinates x y; mn is a section of the bolster distant x from O;

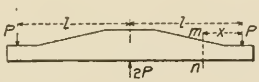

Fig. 4.

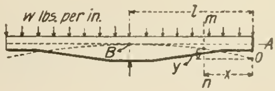

Fig. 1.

*The following is an abstract of a paper which is given in complete form in the Railroad Gazette for December 23, 1898.