where u and v are co-ordinates of points on the pseudo-sphere and u' and v' co-ordinates of points on S. The equations of transformation from S to the plane are

$$v = x$$

$$c\,e^{-\frac{u}{d}} = y$$

The real part of the surface being represented on the strip included between $y = c$ and $y = c\,e$.

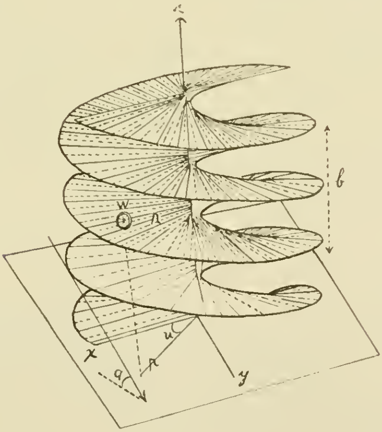
Comparison of Gauss' and Cayley's Proofs of the Existence Theorem.

O. E. Glenn.

[By title.]

Motion of a Bicycle on a Helix Track.

O. E. Glenn.



The equation of the helix surface may be conveniently expressed in surface co-ordinates, thus:

$$x = r \cos u \equiv f_1(ru)$$

$$y = r \sin u \equiv f_2(ru)$$

$$z = \frac{bu}{2\pi} \equiv f_3(ru)$$

in which r represents the distance of a point from the z axis, and u the angle between the x axis and the projection of r upon the (xy) plane; b being a constant.

It will be assumed here that there is a force of friction equal and opposite to the centrifugal force, of a particle (or wheel) moving down the surface, under the action of gravity (g). If these equal and opposite vectors be introduced, the problem reduces to that of determining the motion of a particle (or wheel) on a fixed smooth surface.

The general equation of kinetic energy* is,

$$(1)\quad d\left(\tfrac{1}{2}mv^2\right)=\left\{X\frac{\dagger df_1}{dr}+Y\frac{df_2}{dr}+Z\frac{df_3}{dr}\right\}dr+\left\{X\frac{df_1}{du}+Y\frac{df_2}{du}+Z\frac{df_3}{du}\right\}du,$$

where m represents the mass, v the velocity and X, Y and Z the axial components of the impressed forces.

Denoting the angle between the [xy] plane and the tangent plane of the surface by a there results:

$$(2)\quad X=mg\sin a\cos a\cos u=mg\frac{\sin 2a}{2}\cos u.$$

$$Y=mg\sin a\cos a\sin u=mg\frac{\sin 2a}{2}\sin u.$$

$$Z=mg.$$

And equation (1) reduces to

$$d\left(\tfrac{1}{2}mv^2\right)=\left[g\frac{\sin 2a}{2}\cos^2 u+g\frac{\sin 2a}{2}\sin^2 u\right]m\,dr.$$

$$+\left[-g\frac{\sin 2a}{2}r\sin u\cos u+g\frac{\sin 2a}{2}r\sin u\cos u+\frac{gb}{2\pi}\right]mdu\text{; or,}$$

$$(3)\quad d\left(\tfrac{1}{2}mv^2\right)=m\left\{g\frac{\sin 2a}{2}\right\}dr+\frac{mgb}{2\pi}du.$$

But the angle a equals,

$$a=\cos^{-1}\frac{2\pi r}{\sqrt{4\pi^2r^2+b^2}}$$

Whence $\dfrac{\sin 2a}{2}=\sin a\cos a=\dfrac{2\pi rb}{4\pi^2r^2+b^2}$ and from (3).

$$(4)\quad d\left(\tfrac{1}{2}mv^2\right)=\left\{\frac{2\pi b\,mgr}{4\pi^2r^2+b^2}\right\}dr+\left\{\frac{mgb}{2\pi}\right\}du.$$

This, upon integration, gives,

$$(5)\quad v^2=\frac{gb}{2\pi}\log\left[\frac{r^2+\frac{b^2}{4\pi^2}}{r_0^2+\frac{b^2}{4\pi^2}}\right]+\frac{gb}{\pi}u,$$ the initial conditions being $v=0$ and $r=r_0$ when $u=0$.

* Ziwet Mechanics, p. 103, Vol. III.

† These are partial derivatives.

Now $v^2 = \left\{\frac{^*df_1}{dr}\frac{dr}{dt} + \frac{df_1}{du}\frac{du}{dt}\right\}^2 + \left\{\frac{df_2}{dr}\frac{dr}{dt} + \frac{df_2}{du}\frac{du}{dt}\right\}^2 + \left\{\frac{df_3}{dr}\frac{dr}{dt} + \frac{df_3}{du}\frac{du}{dt}\right\}^2$

in which t represents the time and v the velocity. Therefore,

(6) $v^2 = \left\{\cos u \frac{dr}{dt} - r \sin u \frac{du}{dt}\right\}^2 + \left\{\sin u \frac{dr}{dt} + r \cos u \frac{du}{dt}\right\}^2 + \left\{\frac{b}{2\pi}\frac{du}{dt}\right\}^2$

$= \left\{\frac{dr}{dt}\right\}^2 + \left\{r^2 + \frac{b^2}{4\pi^2}\right\}\left\{\frac{du}{dt}\right\}^2.$

From (5) and (6).

(7) $\frac{gb}{2\pi}\log\left\{\frac{r^2 + \frac{b^2}{4\pi^2}}{r_0^2 + \frac{b^2}{4\pi^2}}\right\} + \frac{gb}{\pi}u = \left\{\frac{dr}{dt}\right\}^2 + \left\{r^2 + \frac{b^2}{4\pi^2}\right\}\left\{\frac{du}{dt}\right\}^2$

This is the differential equation of the motion.

Its integral furnishes solutions of the following:

1. What is the time of descent?
2. What is the equation of the curve of quickest descent?
3. What is the space passed over in a given time?
4. What is the velocity at any instant?
5. What is the normal pressure on the surface?

Problem: A wheelman rides down a helix surface along the line of pitch 30°, keeping his wheel at a constant radial distance of 30 feet. Find the time of descent and his velocity upon reaching the ground; the helix making one complete turn.

Since r is constant and equal to r_0, we have:

(8) $r = r_0 = 30.$

$b = 2\pi r \tan 30^0 = 3.1416 \times 60 + \frac{1}{3}\sqrt{3} = 108.824.$

$g = 32.$

Equation (7) now becomes,

$$\left\{r_0 + \frac{b^2}{4\pi^2}\right\}\left\{\frac{du}{dt}\right\}^2 = \frac{gb}{\pi}u$$

Substituting from (8)

$$\frac{du}{dt} = \left\{\frac{32 \times 108.82}{3.1416 \times 1199.982}\right\}^{\frac{1}{2}}\sqrt{u} = .96\sqrt{u}.$$

$$\therefore t = \frac{2}{.96}\sqrt{u}\Big]_0^{2\pi} = \frac{200}{96}\sqrt{2 \times 3.1416} = 5.2 \text{ seconds} = \text{time of descent.}$$

From equation (4).

$v = \sqrt{\frac{32 \times 108.824}{3.1416}}\sqrt{u} = \sqrt{64 \times 108.824} = 83.4$ ft. per second = velocity at bottom.

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* Partials.

It may be observed that the velocity is the same as that acquired by a body falling through the height h, and is independent of the radial distance, r. The time of descent is directly proportional to r; and both are independent of the weights. That is, we have the theorem:

Motion on the helix surface is equivalent to that on the incline plane, when r is constant.

A Generalization of Fermat's Theorem.

Jacob Westlund.

Consider the function

$$(1)\quad F_{(\alpha,\ A)} = \alpha^{n(A)} - \left(\alpha^{\frac{n(A)}{n(P_1)}} + \dots + \alpha^{\frac{n(A)}{n(P_i)}}\right) + \left(\alpha^{\frac{n(A)}{n(P_1P_2)}} + \dots\right) - \dots + (-1)^i \alpha^{\frac{n(A)}{n(P_1P_2\dots P_i)}},$$

where α is any algebraic integer and A any ideal in a given algebraic number field, $P_1, \dots P_i$ are the distinct prime factors of A, and n(A) denotes the norm of A. The theorem which we shall prove is that $F(\alpha, A)$ is always divisible by A.

For the case when α and A are rational integers several proofs of the divisibility of $F(\alpha, A)$ by A have been given*.

When A is a prime ideal the function $F(\alpha, A)$ reduces to $\alpha^{n(A)} - \alpha$, which, as we know, is divisible by A.

Let us first consider the case when $A = P_1^{s_1}$, where P_1 is a prime ideal of degree f, and p_1 the rational prime divisible by P_1. Then

$$F(\alpha,\ P_1^{s_1}) = \alpha^{p_1^{fs_1}} - \alpha^{p_1^{f(s_1-1)}}.$$

But

$$\alpha^{p_1^{f(s_1-1)}(p_1^f-1)} \equiv 1, \bmod P_1^{s_1}$$

and

$$\alpha^{p_1^{fs_1}} \equiv \alpha^{p_1^{f(s_1-1)}}, \bmod P_1^{s_1}$$

hence

$$(2)\quad F(\alpha,\ P_1^{s_1}) \equiv 0, \bmod P_1^{s}.$$

Now, suppose $A = B \cdot P_1^{s_1}$ where B is any ideal not divisible by P_1. Then we can easily derive the following relation:

$$F(\alpha^{p_1^{fs_1}},\ B) - F(\alpha,\ B \cdot P_1^{s_1}) = F(\alpha^{p_1^{f(s_1-1)}},\ B),$$

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* Dickson, Annals of Mathematics, 2d Series, Vol. 1, 1899, p. 31.