A Theorem on Addition Formulae.

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The theorem stated here is a corollary of a general theorem on a certain class of functional equations, whose theory has not been completed at the time of writing.

Abel has shown that if a function, $\phi(x, y)$, has the property:

$\phi[z, \phi(x, y)]$ is a symmetrical function of x, y, and z; then there exists another function such that:

$$f(x) + f(y) = f[\phi(x, y)].$$

The corollary mentioned proves the converse of this theorem, and shows further, that a necessary and sufficient condition for the solution of an addition formula in the form:

$$f(x) + f(y) = f[z(x, y)],$$

where $z(x, y)$ is supposed given as a known function of x and y, is that the ratio:

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}$$

shall assume the form of the ratio of a function of x alone, to a function of y alone, both of which functions have an indefinite integral, possessing each an inverse function, viz:

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{u'(x)}{u'(y)}$$

Furthermore, if we designate the inverse function by the bar,

$$z(x, y) = \bar{u}[u(x) + u(y)]$$

is another necessary and sufficient restriction on the function $z(x, y)$.

If the equation be given in the form:

(2) $$z[f(x), f(y)] = f(x + y),$$

the necessary and sufficient conditions are:

$$\frac{\dfrac{\partial z}{\partial s}}{\dfrac{\partial z}{\partial t}} = \frac{u'(s)}{u'(t)} \qquad \begin{array}{l} s = f(x). \\ t = f(y). \end{array}$$

$$z(s, t) = \bar{u}\,[u(s) + u(t)].$$

The solution for the unknown function in (1), under the restrictions named above is

$$f(x) = \lambda\, u(x), \qquad \lambda = \text{arbritrary constant},$$

and for (2) is

$$f(s) = \lambda\, u(s), \text{ or as before; } f(x) = \lambda\, u(x).$$

It will be further noticed that if

$$z\,[w, z(x, y)] = \text{symmetric function},$$

then

$$f(x) + f(y) = f\,[z(x, y)], \text{ by Abel's theorem.}$$

We prove the converse. Necessarily

$$z(x, y) = \bar{u}\,[u(x) + u(y)].$$

$$z\,[w, z(x, y)] = \bar{u}\,[u(w) + u\{\bar{u}\,(u(x) + u(y))\}] = \bar{u}\,[u(w) + u(x) + u(y)],$$

which is a symmetric function.

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