A New Problem in Hydrodynamics with Extraneous Forces Acting.

By Edward Lee Hancock.

The solution of most problems in hydrodynamics depends upon the proper combination of the equations of motion of the fluid interior of a given closed surface with the differential equation of the surface, or with the equations expressing the boundary conditions.

Lord Kelvin has shown that the differential equation of the surface for both compressible and incompressible fluids has the following form:

$$u.F'(x) + v.F'(y) + w.F'(z) + F'(t) = 0$$

where (t) is a variable parameter of the equation

$$F(x, y, z, t) = 0.$$

In the treatment of problems of the motion of incompressible fluids in three dimensions, where the surface under discussion is spherical or nearly so, the usual particular solutions of Laplace's equation ($\nabla^2 \phi = 0$), such as, zonal, tesseral and spherical harmonics, are adequate, since in these cases the velocity-potential satisfies Laplace's equation. The solution used in any particular case depends upon the symmetry of the boundary conditions. Where the surface differs much from the spherical form as in ellipsoids, ellipsoidal harmonics are used. Problems of this kind have been extensively investigated.

In discussing the anchor ring Mr. W. M. Hicks[1] has derived modified forms of the zonal, tesseral and spherical harmonics by means of which the potential both outside and inside the ring may be completely investigated. The same problem has been solved by Mr. F. W. Dyson[2] by using elliptic integrals.

The problem is much simplified when the motion takes place in a single plane, in which case, if the boundary consists of a straight line, two parallel straight lines, or is rectangular, the velocity-potential may be expressed as a Fourier's series or a Fourier's integral.

1. Phil. Trans. 1893.
2. Phil. Trans. 1881, Part III.

In other cases there is no direct method of procedure. The inverse process of finding what boundary conditions will give known solutions of Laplace's equation is used, with the hope of finding the desired solution. The method of images is also applicable to some cases, more especially perhaps in the case of rotational motion.

For the irrotational motion of a perfect liquid there always exists a velocity-potential which satisfies the equation

$$\nabla^2\varphi = 0.$$

The potential φ and the rectangular velocities u, v and w may be found from the given conditions, for all points of the interior. The potential being always least at the boundary the lines of flow and equipotential lines begin and end there. This is true whether the motion is "steady" or not and true, therefore, when the extraneous force is gravity.

Much work has been done on the motion of many of the regular solids immersed in a liquid, when acted upon by a system of impulsive forces and also by constant forces. The motions of the liquid in the neighborhood of such solids has also been discussed. Both tidal waves and waves due to local causes have been investigated and their properties discussed to some extent. The related problem of the effect of high land masses upon neighboring bodies of water has been worked out by Professor R. S. Woodward and others.

Perhaps the most familiar problem of the effect of an extraneous force upon a body of liquid, is the "Torricelli Theorem" on the efflux of a liquid from an aperture in the side or bottom of the containing vessel. There the vessel is kept filled to a constant level the motion becomes steady making $\frac{du}{dt} = 0$, $\frac{dv}{dt} = 0$ and $\frac{dw}{dt} = 0$; and giving the well-known result $q^2 = 2\,gz$, where q is the velocity. In case the liquid rotates under the influence of gravity angular velocity is introduced, giving $\frac{dv}{dx} - \frac{du}{dy} = 2w$. Showing that a velocity potential does not exist, and that such motion could not take place in a perfect liquid.

Cases of motion where no extraneous forces are acting have been completely worked out by methods of conjugate functions and the theory of images. iIn these cases the lines of flow and equipotential lines are orthogonal systems of curves, and methods of plotting such are easily devised. But when extraneous forces are acting these lines no longer belong to orthogonal systems of curves and no method has yet been devised by means of which the lines could be drawn under specified conditions.

It was hoped that some graphical method applicable to all cases might be found in connection with the present work, but thus far none has been discovered that is at all general. I have found the equipotential lines and lines of flow for a rectangular area where a constant extraneous force is acting.

Taking the liquid as incompressible since the external forces is constant the motion is steady and the velocity potential may be made to satisfy the equation

$$\frac{\delta^2\phi}{dx^2} + \frac{\delta^2\phi}{dy^2} = 0$$

$$\text{and} \frac{\delta\phi}{\delta x} = ku, \quad \frac{\delta\phi}{\delta z} = kw.$$

A constant must be added to one of these velocities to express the effect of the constant force. This is more clearly seen perhaps in the case of vertical motions due to the force of gravity. In this case the constant to be added to w is of course g and since this is a constant Laplace's equation is still satisfied. The lines of flow and equipotential lines are no longer orthogonal, but are, as we shall presently see, inclined at different angles, being tangent at some points of the interior.

If the area be taken in the sphere of attraction of the earth and near enough so that the attraction may be taken as constant we shall have

$$u = k\,\frac{\delta\phi}{dx}$$

$$v = k\,\frac{\delta\phi}{dz} + k\rho g.$$

where ϕ satisfies Laplace's equation.

Professor C. S. Slichter[1] has shown that the motions in an area A B C D, Fig. 1, filled with sand and having water flowing through it, entering along A B and flowing out along A D—the sides B C and C D being impervious—may be fully discussed by replacing the sand and water by a perfect liquid having a velocity potential, and that the velocity potential in this case would be identical with the pressure function. This being true, it is possible to find the pressure at any point in the interior as well as the component velocities at these points, just as soon as the

1. 19th Annual Report, U. S. Geological Survey, Part II.

boundary conditions are known. Accordingly in what follows the velocity potential will be replaced by the pressure function.

If the section be horizontal, the problem may be treated in the usual way, but in case the section is vertical the extraneous force, gravity, gives a system of curves which are not orthogonal.

Let D C = a and A D = b, and suppose the head of water along A B zero. The boundary conditions then to be satisfied are:

$$P = 0 \text{ when } x = 0$$
$$P = 0 \text{ when } x = a$$
$$P = h \text{ when } z = b$$
$$w = 0 \text{ when } z = 0$$

And since the area is a rectangle P, u and w are expressed as Fourier's series:

$$P = \frac{4g\rho a}{\pi^2} \sum_{n=1}^{n=\infty} \frac{\sinh \frac{n\pi(b-z)}{2a}}{n^2 \cosh \frac{n\pi b}{2a}} \cdot \sin \frac{n\pi x}{2a}$$

This differentiated with respect to x and z for u and w gives:

$$u = \frac{4g\rho k}{\pi} \sum_{n=1}^{n=\infty} \frac{\sinh \frac{n\pi(b-z)}{2a}}{n \cosh \frac{n\pi b}{2a}} \cdot \cos \frac{n\pi x}{2a}$$

$$w = \frac{4g\rho k}{\pi} \sum_{n=1}^{n=\infty} \frac{\cosh \frac{n\pi(b-z)}{2a}}{n \cosh \frac{n\pi b}{2a}} \cdot \sin \frac{n\pi x}{2a} + g\rho k$$

In the above equations *n* represents each of the successive odd numbers, a and b being the sides of the rectangle may have any desired value. But for simplicity they were in the present case taken equal to ten, and for the same reason $g\rho k$ was taken equal to unity.

Making these changes the equations become:

$$P = \frac{80}{\pi^2} \sum_{n=1}^{n=\infty} \frac{\sinh \frac{n\pi(10-z)}{20}}{n^2 \cosh \frac{n\pi}{2}} \cdot \sin \frac{n\pi x}{20}$$

$$u = \frac{4}{\pi} \sum_{n=1}^{n=\infty} \frac{\sinh \frac{n\pi(10-z)}{20}}{n \cosh \frac{n\pi}{2}} \cdot \cos \frac{n\pi x}{20}$$

$$w = \frac{4}{\pi} \sum_{n=1}^{n=\infty} \frac{\cosh \frac{n\pi(10-z)}{20}}{n \cosh \frac{n\pi}{2}} \cdot \sin \frac{n\pi x}{20} + 1$$

From these equations the values of P, u and w were found at each of the one hundred points given in the area. This was done by computing the series for $x=1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ when $z=1$, and then when $z=2, 3, 4, 5, 6, 7, 8, 9, 10$, i. e., by making one hundred computations of each series. The value of u and w being found for each point it was not difficult to determine the resultant in both magnitude and direction. This gave the flow at each of the points of the area. We find from Fig. 1 that there is actual motion throughout the whole area.

The motion, indeed, at some points is very slight, but there is no point in the entire area where there is no motion. This is important if we regard this as an immense area in homogeneous ore-bearing rock. It indicates that at every point of the area the water is continually moving and coming into contact with new rock surfaces, thus increasing its capacity for dissolving the mineral salts from the area. From the length and direction of the arrows it is seen that at the corner D the lines are crowded down closer together than at A. This shows that the constant force gravity has distorted the field, causing the lines of flow to be concentrated at the bottom, and showing that underground waters must take very long journeys before reaching their destination and so come in contact with a very great area of rock surface.

As before stated, the relations of the equipressure lines to the lines of flow differ from that found in horizontal planes. From Fig. 1 it is seen that the angle between the systems of curves varies from nearly a right angle to two right angles, that is, to tangency. In fact, there is in the area what may be called a line of tangency meeting the sides A D and D C. These lines of flow as before indicated taken at equal distances along A B crowd near each other down near D, showing the effect of gravity upon them. If we cause the constant force g to cease to act in the case under consideration, the lines of flow would be arcs of circles cutting A B and A D at equal distances from A. The effect of gravity then is to pull these arcs of circles out into cycloidal-like curves crowding near D C. As a matter of fact the curve drawn from $x=5$, $z=10$ is nearly a cycloid. Those in the upper left-hand corner being too low and long and those in the lower right-hand corner too short and high for cycloids.

The lines of pressure are hyperbola-like curves drawn for pressures, 1, 2, 3, 4, etc., all the curves beginning and ending in the boundary.

It is easy to see that we may take a similar area a b to the right of A B C D and leaving an open face similar to A D and an impervious bottom and water at zero pressure along the top. We should then have these two areas one on each side of B C with the liquid flowing in opposite directions. The liquid in each area flows directly down B C and so the motion will not be interrupted if B C be removed. That is, the method of images is applicable horizontally. If, however, a similar area to A B C D be taken just below C D we can not say that the method of images as usually applied holds true. We may regard A D in the upper area as an absorbing slit and A D in the lower area as a similar slit and the position C D between them as a mirror the corresponding parts of A D in the upper and lower slits are not found at equal distances above and below C D. They are found drawn down by gravity so that the method of images must be modified for vertical distributions. By integrating u with respect to z between the limits b and $\frac{9}{10}\cdot b$; $\frac{9}{10}$ b and $\frac{8}{10}\cdot b$, etc., the amount of flowage from each of the ten equal divisions of A D may be calculated. And in a similar way the amount of liquid going in at each of the ten equal divisions of A B is obtained by integrating w with respect to x between the limits a and $\frac{9}{10}\cdot a$; $\frac{9}{10}\cdot a$ and $\frac{8}{10}\cdot a$, etc. The equations for the flowage and the amount absorbed are then:

$$f=\int_c^d u\,dz=\frac{8ag\rho k}{\pi^2}\sum_{n=1}^{n=\infty}\frac{\cosh\frac{n\pi(b-z)}{2a}}{n^2\cosh\frac{n\pi b}{2a}}\cdot\cos\frac{n\pi x}{2a}\Big]_c^d$$

$$a=\int_c^d w\,dx=\frac{8ag\rho k}{\pi^2}\sum_{n=1}^{n=\infty}\frac{\cosh\frac{n\pi(b-z)}{2a}}{n^2\cosh\frac{n\pi b}{2a}}\cdot\cos\frac{n\pi x}{2a}\Big]_c^d\ g\rho kx\Big]_c^d$$

where c varies from $\frac{9}{10}$. b or $\frac{9}{10}$. a down to zero, and d varies from a or b down to $\frac{1}{10}$. b or $\frac{1}{10}$. a. Solving the ten equations for the ten different values of f and a, we get the following table:

No....	1	2	3	4	5	6	7	8	9	10
a	.958	.875	.800	.726	.664	.611	.566	.535	.512	.502
f	.042	.126	.216	.315	.424	.556	.716	.935	1.24	2.07

TABLE I.

It will be seen from the table by counting the divisions from A as 1, 2, 3, etc., that nearly half the water flows through the first three divisions and that there is a gradual decrease toward B. The relative value of f from the different divisions shows a very slight flowage from the first division with a rapid increase from each of the succeeding divisions until the two lower divisions at D carry off one-half of the amount absorbed. This shows in a very vivid way the pronounced effect of gravity or any constant external force upon a liquid. The amount going in along A B is of course equal to the amount flowing out along A D, since the equation of continuity must hold true.

It is interesting to note that the curve given by plotting the flowage from A D is very nearly a tractrix or antifriction curve. See Fig. 3. It would undoubtedly be an exact tractrix had the number of divisions of A D been taken small enough, i. e., if twenty or thirty equal divisions had been taken instead of ten.

In Fig. 3 the line O X corresponds to the distance A D in Fig. 1, and the y-coördinates of the curve are given by the values f taken from Table I.

Fig. 4 shows the distribution of absorption into the area A B C D along A B, the line A B of the figure corresponding to the line A B of the area. The y-coördinates of the curve being taken from Table I as the different values of a.

Figs. 3 and 4 then show the distribution of absorption and flowage along A B and A D.

Extending this method by taking A B one hundred and keeping A D ten, we get approximately an artesian well area. The values of f and a for this case are given below:

No....	1	2	3	4	5	6	7	8	9	10
a	5.51	1.44	.139	.044	.028	....	....	....	....	.005
f	.040	.162	.210	.348	.446	.616	.762	.981	1.32	2.53

TABLE II.

It will be seen that the amount flowing in at the first division of A B is about two-thirds the total amount flowing into the entire area, and that this supplies the flowage for the first nine divisions of A D while the tenth division of A D gives out the water from $\frac{9}{10}$ the distance A B. If the rock in the area be soluble it is easily seen that the water flowing from this lowest division of A D will be very highly charged with mineral matter, while the remaining two-thirds that flows out above will be very slightly charged. This is more especially evident when the long sweeping paths of the water are considered compared with the very short paths of the waters of the first division of A B. We have this represented graphically in Fig. 5, where the lines of flow are drawn for the case where A B = 100 and A D = 10, or a typical artesian area. If A D be a crevice in the rock it is evident that this place will be favorable for the deposition of the mineral salt dissolved in the water since the pressure is released at this point and there is apt to exist some reagent that will cause a precipitate of the ore. This reagent may exist in the crevice itself or in the opposite wall.

In Fig 6 the curve has been plotted for the flowage from A D for the case A D = 10 and A B = 100. This does not differ much from the case where A D = 10 and A B = 10, except that the convexity downward is somewhat more pronounced, making the curve less like the tractrix.

Ten equal divisions were taken along A D and the values of y taken from Table II corresponding to different values of f.

The absorption curve for the case A D = 10 and A B = 100 is given in Fig. 7. Here the scale has been somewhat changed due to the large value of A B. The distance A B was divided into one hundred equal divisions, while the same vertical scale was used for y as in the preceding cases. The values of y were taken from Table II, being the different values of a in that table.

The rapid fall of the curve at first and then more gradual fall corresponds to the values of a found in Table II and also emphasizes the relative slowness of the motion of the water in the right-hand half of the area A B C D, Fig. 5, as compared with that of the left-hand half.

The method used in the preceding cases might be extended to areas of different dimensions, but the results would not differ much from those already stated.

If A B be taken greater than one hundred, while A D remains ten, or if we have any similar relation between the two, it will be more advantageous to use the Fourier's integral instead of the Fourier's series, since for such a difference between A B and A D the area may be considered as an infinite strip.

The results obtained are especially interesting in connection with the motion of ground water, because of their bearing on the theory of ore deposits, artesian wells and drainage flumes. The fact that sand through which water is flowing, as before indicated, can be replaced by an ideal liquid having a velocity-potential which is identical with the pressure opens a new field of investigation in hydrodynamics from which many important results will be obtained.

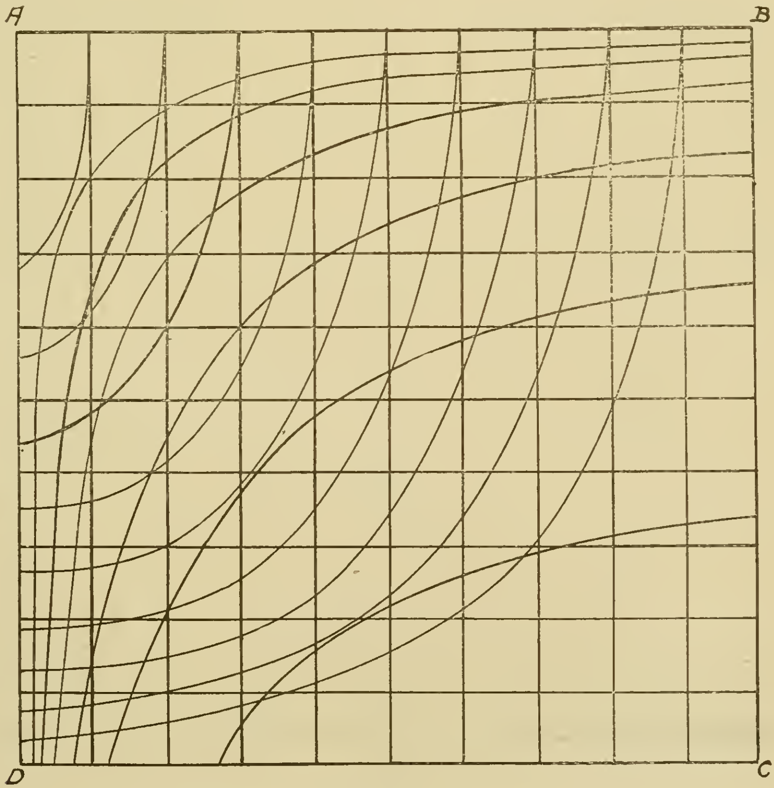

Fig.-1

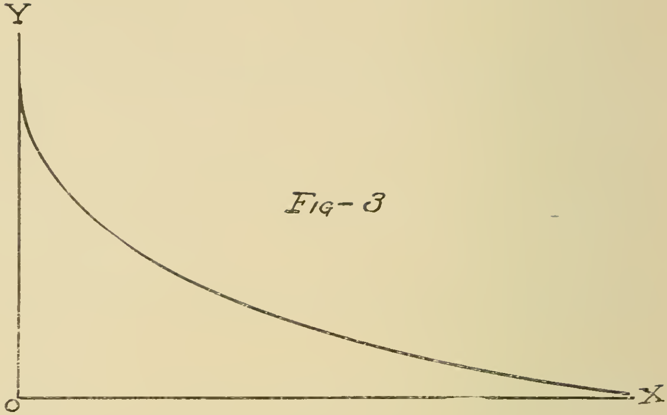

Fig- 3

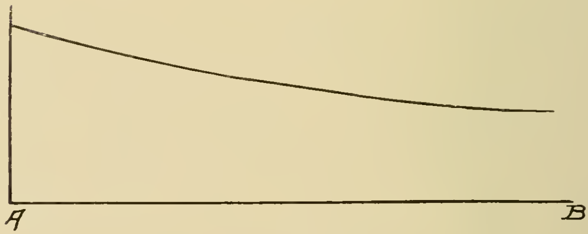

Fig. — 4

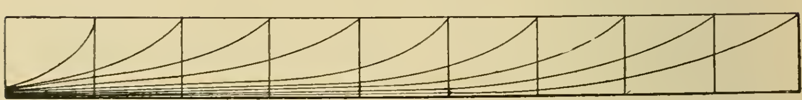

Fig.—5

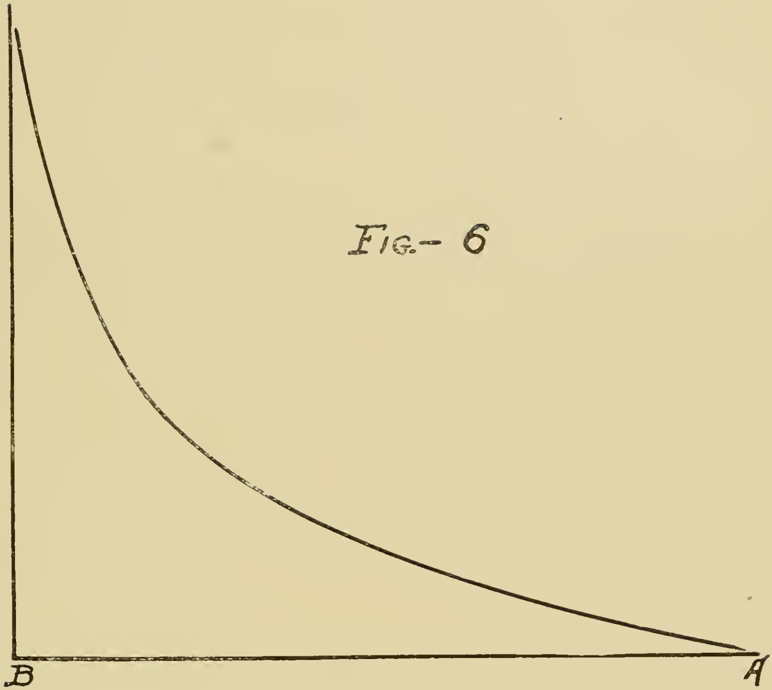

FIG.- 6

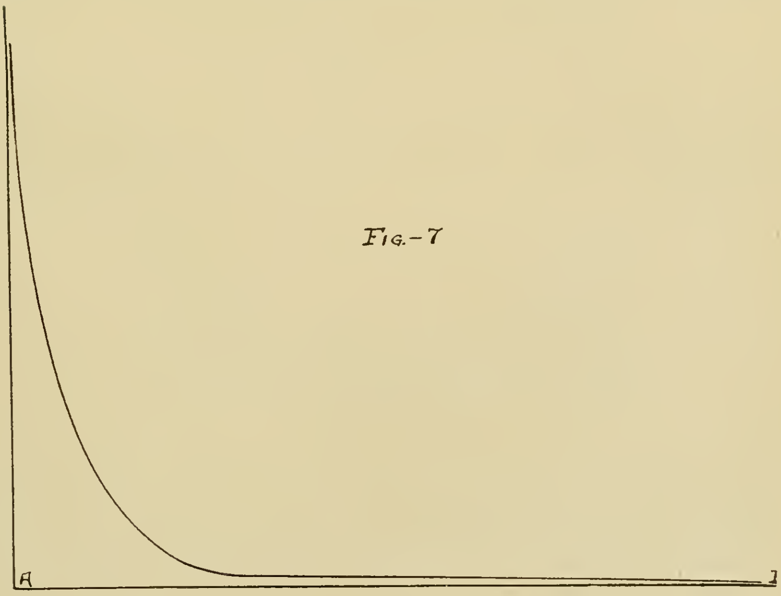

FIG.-7