A Note on the Intersection of Osculating Planes to the Twisted Cubic Curve.

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On page 15 of his Differential Geometry Eisenhart proposes an exercise to show that on any plane there is one and only one line through which can be drawn two osculating planes to the twisted cubic.

§1.

The twisted cubic

$$x = \frac{a_0 + a_1 t + a_2 t^2 + a_3 t^3}{d_0 + d_1 t + d_2 t^2 + d_3 t^3}$$

$$y = \frac{b_0 + b_1 t + b_2 t^2 + b_3 t^3}{d_0 + d_1 t + d_2 t^2 + d_3 t^3}$$

$$z = \frac{c_0 + c_1 t + c_2 t^2 + c_3 t^3}{d_0 + d_1 t + d_2 t^2 + d_3 t^3}$$

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{vmatrix} \neq o \quad \text{(since the curve is twisted)}$$

is carried over by the nonsingular linear transformation:

$$x = \frac{a_1 x' + a_2 y' + a_3 z' + a_0}{d_1 x' + d_2 y' + d_3 z' + d_0}$$

$$y = \frac{b_1 x' + b_2 y' + b_3 z' + b_0}{d_1 x' + d_2 y' + d_3 z' + d_0}$$

$$z = \frac{c_1 x' + c_2 y' + c_3 z' + c_0}{d_1 x' + d_2 y' + d_3 z' + d_0}$$

into the cubic

$$x' = t \qquad y' = t^2 \qquad z' = t^3;$$

planes, straight lines, and points, go over into planes, straight lines, and points, respectively, and in particular, osculating planes go into osculating planes.

The equation of the osculating plane to the cubic

$$x = t \qquad y = t^2 \qquad z = t^3$$

at the point whose parameter value is t, is

$$3t^2 x - 3ty + z - t^3 = 0$$

There is no line in space through which pass three such planes:

$$3t_1^2x - 3t_1y + z - t_1^3 = 0$$
$$3t_2^2x - 3t_2y + z - t_2^3 = 0 \qquad t_1,\ t_2,\ t_3, \text{ all different,}$$
$$3t_3^2x - 3t_3y + z - t_3^3 = 0$$

for the determinant of the coefficients of x, y, z, is equal to

$$9\,(t_1 - t_2)\,(t_2 - t_3)\,(t_3 - t_1) \neq 0$$

and therefore the three planes are linearly independent.

§2.

Given a real* plane

E: $\qquad ax + by + cz + d = 0 \qquad$ a, b, c, not all zero

and the cubic

K: $\qquad x = t \qquad y = t^2 \qquad z = t^3$

The equations of the planes which osculate K at any two distinct points $P_1\,(t_1)$, $P_2\,(t_2)$, $t_1 \neq t_2$, determine a line

L: $\qquad x = s/3 + u \qquad y = p/3 + su \qquad z = 3pu$†

where $s = t_1 + t_2$, $p = t_1\,t_2$, and u is a parameter.

That L lie on E, it is necessary and sufficient that

(1) $$a\,s + b\,p + 3\,d = 0$$
$$b\,s + 3cp + a = 0$$

Write the matrix of the coefficients of equations (1)

M: $$\begin{Vmatrix} a & b & 3d \\ b & 3c & a \end{Vmatrix}$$

and set

$$A = \begin{vmatrix} a & b \\ b & 3c \end{vmatrix}, \qquad B = \begin{vmatrix} a & 3d \\ b & a \end{vmatrix}, \qquad C = \begin{vmatrix} b & 3d \\ 3c & a \end{vmatrix}$$

§3.

Suppose $A \neq 0$. Equations (1) have the unique solution:

$$s = C/A \qquad p = -B/A$$

whence t_1 and t_2 are the roots of the quadratic equation

(2) $$A\,t^2 - C\,t - B = 0$$

Therefore if two distinct planes osculate K and intersect on E, (in case $A \neq 0$), it is necessary that $4\,A\,B + C^2 > 0$, and that the parameter values of their points of osculation be the roots of the quadratic (2).

This condition is also sufficient, for if $A \neq 0$, and if $4\,A\,B + C^2 > 0$ equation (2) determines two real numbers t_1 and t_2, and if we set $s = t_1 + t_2$, $p = t_1\,t_2$, these numbers satifsy equations (1), and the line $x = s/3 + u$,

*This problem is treated throughout as a problem in Geometry, not one in Algebra.

†These equations hold even if one of t_1, t_2, is zero.

$y = p/3 + su$, $z = 3pu$, lies on E and is the intersection of two planes which osculate K. Moreover, since under these conditions, t_1, t_2, s, p, are uniquely determined, there is no other line on E through which pass two planes which osculate K.

If $4\,A\,B + C^2 \leqq 0$, equation (2) has one real root, or no real root, and there exists on E no line through which can be drawn two planes which osculate K.

§4.

Suppose $A = 0$ but B and C are not both zero. Equations (1) have no solution. There is no line on E through which pass two osculating planes.

The results of §3 and §4 may be combined into a theorem:

If not all the determinants of M vanish, there is exactly one line on E, or no line on E, through which pass two osculating planes, according as the equation $A\,t^2 - C\,t - B = 0$ has or has not two real roots.

§5.

Suppose all the determinants of M vanish. Under these conditions E itself osculates K; for, in order that the equation $a\,t + b\,t^2 + c\,t^3 + d = 0$ have three equal linear factors, it is necessary and sufficient that $A = B = C = 0$.

The plane $z = 0$ osculates K at the origin. If E osculates K, the point of osculation is $(-a/b,\ a^2/b^2,\ -a^3/b^3)$ if $b \pm 0$, but $(0, 0, 0)$ if $b = 0$.

The number of osculating planes which can be drawn to K from a point P (x, y, z) is equal to the number of real roots of the equation in t

(3) $$t^3 - 3xt^2 + 3yt - z = 0$$

Write down the matrix*

$$M' : \quad \left\|\begin{matrix} 1 & -x & y \\ -x & y & -z \end{matrix}\right\|$$

and set

$$A' = \begin{vmatrix} 1 & -x \\ -x & y \end{vmatrix}, \qquad B' = \begin{vmatrix} 1 & y \\ -x & -z \end{vmatrix}, \qquad C' = \begin{vmatrix} -x & y \\ y & -z \end{vmatrix}$$

then the discriminant of (3) is

$$D = 3\begin{vmatrix} 2\,A' & B' \\ B' & 2\,C' \end{vmatrix}$$

The points of the plane E may be classified as follows:

(1) Suppose at a point P of E, $D > 0$.

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*Reduced from $\left\|\begin{matrix} 3 & -6\,x & 3\,y \\ -3\,x & 6\,y & -3\,z \end{matrix}\right\|$

Equation (3) has three real roots, t_1, t_2, t_3; one of these t_1 say, determines E itself; the other two determine a pair of oscultating planes:

$$3\,t_2^2\,x - 3\,t_2\,y + z - t_2^3 = 0$$

$$3\,t_3^2\,x - 3\,t_3\,y + z - t_3^3 = 0$$

distinct from E and from each other; their intersection does not lie on E, else would the three oscultaing planes be linearly dependent. Therefore, these two planes cut out from E a pair of lines intersecting in P, through each of which passes a pair of osculating planes. E itself and one other.

(2) Suppose at a point P of E, $D = 0$ but A', B', C', are not all zero.

Equation (3) has only two *roots, both real; one of these determines E and the other determines an osculating plane distinct from E which intersects E in a line through P.

(3) Suppose at a point P of E, $A' = B' = C' = 0$; then is $D = 0$.

There is in fact only one point on E at which $A' = B' = C' = 0$, for from these equations follow $x=x$, $y = x^2$, $z = x^3$; therefore P is on K and is therefore the point of osculation of E and K. Under these conditions equation (3) has only one† root and that determines E.

(4) Suppose at a point P of E, $D < 0$.

Equation (3) has only one real root and that determines E.

These results may be combined into a theorem:

If all the determinants of M vanish, E itself osculates K. Through every point of E at which $D>0$ there may be drawn a unique pair of lines on E, through each of which pass two osculating planes; through every point of E at which $D=0$ (except the point where E osculates K) there may be drawn a unique line on E through which pass two osculating planes; through every other point of E (including the point of osculation) there exists no line on E through which pass two osculating planes.

§6

Examples:

1. E: $3\,x + 3\,y - 2\,z - 5 = 0$

M: $\begin{Vmatrix} 3 & 3 & -15 \\ 3 & -6 & 3 \end{Vmatrix}$, $A = -27$, $B = 54$, $C = -81$.

(2) $t^2 - 3\,t + 2 = 0$ $\qquad t_1 = 1$, $t_2 = 2$, $s = 3$, $p = 2$.

L: $x = 1 + u$, $y = 2/3 + 3\,u$, $z = 6\,u$.

Through L pass the two osculating planes:

$3\,x - 3\,y + z - 1 = 0$, $\qquad 12\,x - 6\,y + z - 8 = 0$.

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*That is, two equal linear factors distinct from the third linear factor.

†That is, three equal linear factors.

2. E: $x + 3y + z = 0.$

M: $\begin{Vmatrix} 1 & 3 & 0 \\ 3 & 3 & 1 \end{Vmatrix}$ $A = -6, \quad B = 1, \quad C = 3.$

(2) $6t^2 + 3t + 1 = 0.$ No real root: no line.

3. E: $x + 2y + z = 0.$

M: $\begin{Vmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \end{Vmatrix}$ $A = -1, \quad B = 1, \quad C = 2.$

(2) $t^2 + 2t + 1 = 0$ Only one root: no line

4. E: $3x + 3y + z - 1 = 0$

M. $\begin{Vmatrix} 3 & 3 & -3 \\ 3 & 3 & 3 \end{Vmatrix}$, $A = 0, \quad B = 18, \quad C = 18$

(2) $t + 1 = 0$ Only one root: no line.

5. E: $3x - 3y + z - 1 = 0$

M: $\begin{Vmatrix} 3 & -3 & -3 \\ -3 & 3 & 3 \end{Vmatrix}$ $A = B = C = 0.$

E osculates K at (1, 1, 1) i. e. where $t = 3/3 = 1$, see §5.

a) Consider the point P (— 2, 1, 10) on E

M′: $\begin{Vmatrix} 1 & 2 & 1 \\ 2 & 1 & -10 \end{Vmatrix}$ $A' = -3, \; B' = -12, \; C' = -22.$

$D = 360$; two lines on E through P;

(3) $t^3 + 6t^2 + 3t - 10 = 0 \qquad t_1 = 1, \; t_2 = -2, \; t_3 = -5$

Lines through P:

L_1: $x = -2 - u, \quad y = 1 + u, \quad z = 10 + 6u$

through which pass osculating planes

$3x - 3y + z - 1 = 0, \qquad 12x + 6y + z + 8 = 0.$

L_2: $x = -2 - u, \quad y = 1 + 4u, \quad z = 10 + 15u.$

through which pass osculating planes

$3x - 3y + z - 1 = 0, \qquad 75x + 15y + z + 125 = 0.$

b) Consider the point P (2, 3, 4) on E

M′: $\begin{Vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \end{Vmatrix}$, $A' = -1, \quad B' = 2, \quad C' = -1, \; D = 0.$

(3) $t^3 - 6t^2 + 9t - 4 = 0, \quad t_1 = 1, \quad t_2 = 4$

Line through P:

L: $x = 2 + u, \quad y = 3 + 5u, \quad z = 4 + 12u,$

through which pass osculating planes

$$3x - 3y + z = 1, \qquad 48x - 12y + z = 64$$

c) Consider the point (0, 0, 1) on E:

$$M': \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} \quad A'=0, \; B'=-1, \; C'=0, \; D=-3. \quad \text{No line.}$$

In case E is an osculating plane different from $z = 0$, and P is on E, $t = -a/b$ is a root of equation (3), which can consequently be depressed to the quadratic

(4) $$(bt)^2 - (a + 3bx)\,bt + a\,(a + 3bx) + 3b^2 y = 0$$

and the number of osculating planes through P which are distinct from E is equal to the number of real roots of this equation which are different from $-a/b$.

d) Consider again the point P $(-2, 1, 10)$ on $3x - 3y + z = 1$

(4) $$t^2 + 7t + 10 = 0 \qquad t_1 = -2 \qquad t_2 = -5$$

both different from 1: therefore two lines as before under a).

e) Consider the point P $(-1, 0, 4)$

(4) $$t^2 + 4t + 4 = 0 \qquad t = -2$$

one root different from 1; therefore one line

$$L: \quad x = -1 - u, \qquad y = u, \qquad z = 4 + 6u$$

through which pass osculating planes

$$3x - 3y + z - 1 = 0, \quad \text{and} \quad 12x + 6y + z + 8 = 0$$

§7.

The case where E is an osculating plane may also be treated geometrically by making use of certain considerations given in a later chapter of Eisenhart's book. The equation of the envelope F of the osculating planes to K is obtained by equating to zero the discriminant D of equation (3):

(5) $$3x^2y^2 + 6xyz - 4x^3z - z^2 - 4y^3 = 0$$

Since $x = t$, $y = t^2$, $z = t^3$, satisfies (5) for all values of t, K itself lies on F; in fact, K is the edge of regression of F.

A given osculating plane E not only touches F, but in general cuts out from F a plane curve H, which passes through the point where E osculates K. Every osculating plane different from E, cuts E in a line tangent to H; conversely through every straight line on E tangent to H passes an osculating plane which is distinct from E.* The curve H divides E into two or more regions throughout each of which D is always positive or always negative and therefore serves to classify the points of E into those through which can be drawn two lines, or one line, or no line respectively, which is the intersection of two osculating planes,

*Unless perchance this line is a part of H, as is the case with the x-axis on the plane $z=0$.

On the osculating plane $z=0$ the curve H consists of the x-axis and the parabola $3x^2 = 4y$. It divides the plane into four regions: R_1, the top half plane; R_2, that part of the lower left quarter plane which is "outside" the parabola; R_3, that part of the lower right quarter plane which is "ouside" the parabola; R_4, that part of the lower half plane which is "inside" the parabola. Throughout the first three regions, $D > 0$, while in the fourth region, $D < 0$; everywhere on H itself of course $D = 0$.

Any other osculating plane $3t^2x - 3ty + z - t^3 = 0$, $t \neq 0$, cuts the plane $z = 0$ in the line $3tx - 3y - t^2 = 0$. This equation represents the one parameter family of lines which envelope the parabola $3x^2 = 4y$. The parameter t, is in fact the slope of these tangents.

From any point in R_{123} two tangents can be drawn to the parabola and through each of these pass two osculating planes; from no point in R_4 can a tangent be drawn to the parabola and through this region of $z = 0$ there passes no line which is the intersection of two osculating planes. Through any point on the parabola itself one and only one tangent can be drawn and (excepting the tangent at the origin) this is the intersection of two osculating planes. Through any point (except the origin) on the x-axis, which is a part of H, two tangents can be drawn to the parabola but one of these is in all cases the x-axis itself, through which passes no osculating plane distinct from $z= 0$. Therefore through any point (except the origin) on the x-axis there passes one line which is the intersection of two osculating planes.

Examples on the osculating plane $z = 0$.

a) Consider the point $(0, -3, 0)$ in R_1

$3tx - 3y - t^2 = 0$ gives $t_1 = 3$ $\quad t_2 = -3$

L_1: $\quad 3x - y - 3 = 0$ through which pass osculating planes

$27x - 9y + z = 27$ and $z = 0$

L_2: $\quad 3x + y + 3 = 0$ through which pass

$27x + 9y + z + 27 = 0$ and $z = 0$

b) Consider the point $(1/2, -3/2, 0)$ in R_1

$3tx - 3y - t^2 = 0$ gives $t_1 = 3$ $\quad t_2 = -3/2$

L_1: $\quad$ same as L_1 under a)

L_2: $\quad 6x + 4y + 3 = 0$ through which pass

$54x + 18y + 4z + 27 = 0$ and $z = 0$

c) Consider the point $(2, 3, 0)$ on the parabola

$3tx - 3y - t^2 = 0$ gives $t = 3$

L: $\quad$ same as L_1 under a)

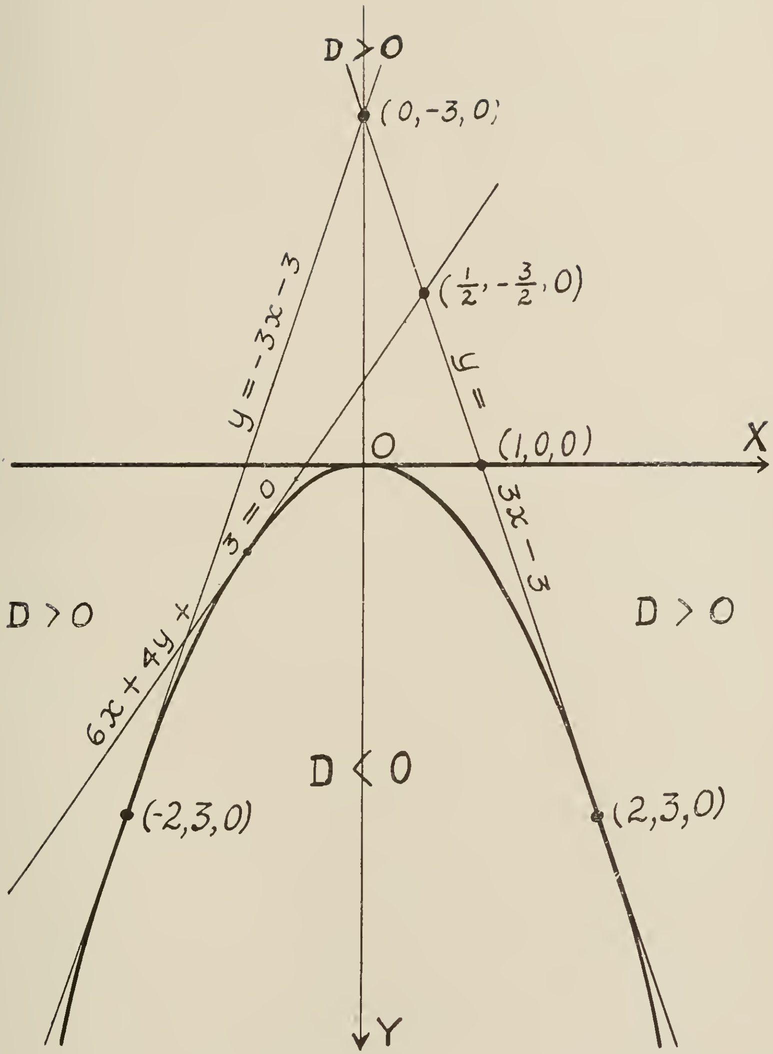
d) Consider the point (1, 0, 0) on the x-axis

$$3\,t\,x - 3\,y - t^2 = 0 \quad \text{gives} \quad t_1 = 0 \quad t_2 = 3$$

$t = 0$ determines the x-axis; $t = 3$ determines the line

L: same as L_1 under a).

These examples are illustrated by the accompanying figure.



Purdue University;
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