Conditions for the Deformation of Surfaces Referred to a Conjugate System of Lines.

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When a surface is subjected to a series of deformations, each form that it assumes during the deformation may be thought of as a separate, distinct surface. We may thus regard a deformation of a surface as a continuous system of surfaces, each representing some form into which the original surface may be deformed. In this paper we consider the problem of determining those surfaces which may be deformed so that a conjugate system of lines will still remain a conjugate system after the deformation is carried out.

We shall suppose that the equations of the surfaces that we consider are given in the form,

$$x = f_1\,(\mu, \nu), \qquad y = f_2\,(\mu, \nu), \qquad z = f_3\,(\mu, \nu),$$

and that the first and second fundamental magnitudes are E, F, G and D, D′, D″, respectively.

If S_2 represents the form that S_1 takes when deformed so that a conjugate system remains a conjugate system, then S_2 is applicable on S_1. But the necessary and sufficient condition that two surfaces should be applicable on each other is that they shall have the same lineal element and the same total curvature.

If the parametric lines, $\mu =$ const., $\nu =$ const., on S_1 and S_2 form a conjugate system, then $D' = o$ for both S_1 and S_2. Since S_1 and S_2 must have the same lineal element and the same curvature, we have from the relation,

$$K = \frac{D\,D''}{EG - F^2}$$

that $D_2 = \lambda\,D_1$ and $D_2'' = \lambda\,D_1''$ where the subscripts refer to S_1 and S_2 respectively, and λ is a function of μ and ν. To determine λ we make use of the fact that Codazzi's equations must be satisfied for both S_1 and S_2. Bianchi* has thus shown that λ must satisfy the equations,

$$\frac{\delta}{\delta\mu}\left(\frac{1}{\lambda}\right) = -\left\{\begin{matrix}12\\2\end{matrix}\right\}'\left(\frac{1}{\lambda} - \lambda\right)$$

*" Vorlesungen über Differential-Geometrie." p. 336.

(1)
$$\frac{\delta}{\delta v}(\lambda) = -\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \left[\lambda - \frac{1}{\lambda} \right]$$

where $\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}'$ and $\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}'$ are the symbols of Christoffel formed with respect to the Gauss sphere. Since now $\frac{\delta^2 \lambda}{\delta u\, \delta v} = \frac{\delta^2 \lambda}{\delta v\, \delta u}$ we have from (1), as the condition of integrability,

$$(2) \quad \lambda^2 \left[\frac{\delta}{\delta v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' - 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \right] = \left[\frac{\delta}{\delta u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' - 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \right]$$

Having given the surface S_1, then to every value of λ which satisfies (1) and (2) there corresponds a surface S_2 of the desired type.

There are three possible cases that may occur under (2). Suppose, first, that the surface S_1 is such that

$$(\mathrm{I}) \quad \frac{\delta}{\delta v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' = 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' = \frac{\delta}{\delta u} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}'$$

In this case the condition of integrability (2) is satisfied for every value of λ, and since equations (1) are of the first order, there are in this case ∞^1 surfaces S_2 which are applicable on S_1 and such that their parametric lines form a conjugate system. We thus have in this case a continuous system of surfaces, and the above equations are the necessary and sufficient condition that a surface may belong to such a system.

Suppose, next, that S_1 is such that

$$(\mathrm{II}) \quad \begin{array}{lcl} \frac{\delta}{\delta v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' = 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' & & \frac{\delta}{\delta v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \neq 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \\ & \text{or,} & \\ \frac{\delta}{\delta u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \neq 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' & & \frac{\delta}{\delta u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' = 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \end{array}$$

In this case λ vanishes or is undefined, and the condition of integrability is not satisfied. Consequently there exists no surface S_2 in this case.

Suppose, finally, that

$$(\mathrm{III}) \quad \begin{array}{l} \frac{\delta}{\delta v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \neq 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \\ \frac{\delta}{\delta u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \neq 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \end{array}$$

We have in this case one, and only one, value for λ^2. If the surface S_1 is such that in addition to (III) being satisfied, (1) are also satisfied, then there is one, and only one, surface S_2 which represents the result of deforming S_1 so that a conjugate system remains a conjugate system after the deformation.

There are two cases which may occur under (III). Suppose that

$$(\mathrm{III_a}) \qquad \frac{\delta}{\delta v}\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' = \frac{\delta}{\delta \mu}\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}'$$

Then $\lambda = \pm 1$ and the surface S_2 is such that its second fundamental magnitudes D_2 and D_2'' are either equal to the corresponding magnitudes D_1 and D_1'' of S_1 or they are the negatives of D_1 and D_1''.

But from the equations (*)

$$\frac{\delta x}{\delta \mu} = \frac{D}{eg - f^2}\left\{ - g\,\frac{\delta X}{\delta \mu} + f\,\frac{\delta X}{\delta v} \right\}$$

$$\frac{\delta x}{\delta v} = \frac{D''}{eg - f^2}\left\{ f\,\frac{\delta X}{\delta \mu} - e\,\frac{\delta X}{\delta v} \right\}$$

Where e, f, g are the fundamental magnitudes of the Gauss' sphere, it is seen that a change in the sign of D and D″ corresponds to a change of sign in the co-ordinates x, y, z of the surface, and therefore the surface S_2 is either identical with S_1 or it is symmetrical to S_1 with respect to a plane or to the origin of co-ordinates.

Suppose next that

$$(\mathrm{III_b}) \qquad \frac{\delta}{\delta v}\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \neq \frac{\delta}{\delta \mu}\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}'$$

In this case there is a unique value of $\lambda^2 \neq 1$. S_1 may therefore be deformed so that after the deformation is carried out the lines $\mu = \text{const.}$, $v = \text{const.}$, form a conjugate system, although they do not form a conjugate system at any time during the deformation. Now, by a theorem of Dini, (**) from relation ($\mathrm{III_b}$) no surface S_0 exists, the spherical images of whose asymptotic lines are the same as the spherical images of a conjugate system of lines on S_1. But from the definition of associate surfaces, there is then no surface to which S_1 is associated, and thus we have the result that when ($\mathrm{III_b}$) is true for any surface S_1 referred to a conjugate system, there exists no surface S_0 to which S_1 is associated.

(*) Bianchi, l. c. p. 134.
(**) Bianchi, l. c. p. 125.