A Family of Warped Surfaces.

C. A. Waldo.

Derivation of the general equation of all warped surfaces having two distinct rectilinear directrices and its application to a few special cases.

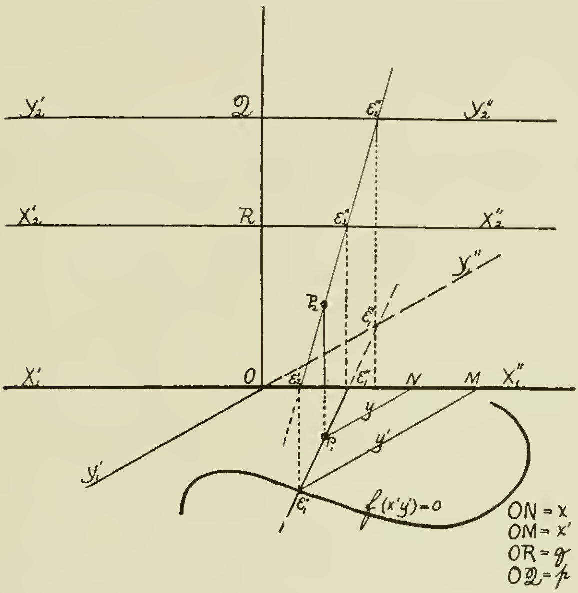

Fig. 1.

Let the surface be defined by the three directrices

$$x = O, \ldots\ldots\ z = p,$$
$$y = O, \ldots\ldots\ z = q,$$
$$f\ (x'y') = O, \ \ldots\ldots z = O.$$

The curve $f\ (x'y') = O$ lies in the plane $z = O$, the X and Y axes are parallel to the rectilinear directrices; the Z axis includes the common perpendicular to the rectilinear directrices, unless otherwise specified.

In the diagram Fig. 1, let $X'\ X''$ be one straight line directrix at the distance q above the plane $z = o$, $Y'\ Y''$ the other at p above $z = o$. Their horizontal projections will be the X and Y axes of reference.

Let (x, y, z) be any point P on the warped surface, and $E' E'' E'''$ the rectilinear element containing it.

Let $O M = x'$, $O N = x$, $O R = q$, $O Q = p$.

Then by similarities and projections the following equations exist:

$$\frac{x'}{x} = \frac{E'''_1 E'_1}{E'''_1 P_1} = \frac{E'''_2 \ E'_2}{E'''_2 \ P_2} = \frac{p}{p-z}, \quad x' = \frac{p x}{p-z}$$

Similarly, $$y' = \frac{q y}{q-z}$$

Substituting these values of $x' y'$ in $f(x' y') = o$, there results the corresponding functional equation,

$$f\left(\frac{p x}{p-z}, \frac{q y}{q-z}\right) = o,$$

which is the equation in Cartesian co-ordinates X, Y axes general, Z axis perpendicular to X and Y of the warped surfaces as defined above and includes *every* warped surface with two distinct rectilinear directrices. For its application it requires that a section of the surface should be known parallel to the right-line directrices and not including either of them. This general surface is referred directly to the orthogonal projections of two warped lines in space upon a plane parallel to both, and to their common perpendicular. The angle at which the lines intersect is implicitly contained in the equation of the surface. The form of the equation of the surface does not change, therefore, when the surface itself is deformed by changing the angle in space of the right line directrices, provided the form of the equation of the plane curve directrix remains unchanged.

It is also at once evident that the method derives immediately the Cartesian equation of the warped surface determined by the fact that an element cuts a curved directrix, a linear directrix and is parallel to a given plane. This is equivalent to saying that one of our parameters p, q, remains finite while the other becomes indefinitely great.

For simplicity suppose the three axes always at right angles to each other unless otherwise specified.

The Hyperbolic Paraboloid.

(a) Let $f(x' y') = x' - y' = O$.

Then $f\left\{\frac{p x}{p-z}, \frac{q x}{q-z}\right\} = \frac{p x}{p-z} - \frac{q y}{q-z} = o$.

Let $p = 1$, $q = -1$

Then $x + xz - y + yz = o$.

Rotate the xy axes through $\pi/4$, then the zx axes in the same way, and there results the well known equation,

$$x^2 - z^2 = 2y.$$

(b) Let $f(x' y') = x' y' - c = o$.

Then $f\left\{\frac{px}{p-z}, \frac{qy}{p-z}\right\} = \frac{px}{p-z}\,\frac{qy}{q-z} - c = \frac{px}{p-z}\,\frac{y}{1-\frac{z}{q}} - c = o.$

Let $p = 1$ and q become indefinitely great,

Then $xy = c\,(1-z)$.

Rotate the zy axes through $\pi/4$, let $c = 1$ and

$1 - z = Z$,

Then $x^2 - y^2 = 2Z$.

Compare this operation and result with the next.

The Hyperboloid of One Sheet.

Let $f(x' y') = x' y' - c = o$

as above $\frac{px}{p-z}\,\frac{qy}{q-z} = c.$

let $p = 1$, $q = -1$

Then $xy = c\,(1 - z^2)$.

Rotate xy axes through $\pi/4$, let $c = 1/2$.

Then $x^2 - y^2 + z^2 = 1$.

A Cubic Surface with Parabolic Sections.

Let $f(x' y') = y'^2 - x' = o$.

Then $f\left\{\frac{px}{p-z}, \frac{qy}{q-z}\right\} = \frac{q^2 y^2}{(q-z)^2} - \frac{px}{p-z} = o.$

a. Let $p = 1$ and $q = -1$. Then

$y^2\,(1-z) = x\,(1 + z)^2$, one of the cubical warped surfaces.

b. Let $p = 1$, $q = \infty$, then $y^2\,(1 - z) = x$.

c. Let $q = 1$, $p = \infty$, then $y^2 = x\,(1 - z)^2$.

Biquadratic Surface with Hyperbolic Sections.

Let $f(x' y') = x'^2 - y'^2 - c = o$

Then $f\left[\frac{px}{p-z}, \frac{qy}{q-z}\right] = \frac{p^2 x^2}{(p-z)^2} - \frac{q^2 y^2}{(q-z)^2} - c = o$

a. Let $p = 1$, $q = -1$, $c = 1$

Then $x^2\,(1 + z)^2 - y^2\,(1 - z)^2 = (1 - z^2)^2$

b. Let $p = 1$, $q = \infty$, $c = 1$

Then $x^2 - y^2 (1 - z)^2 = (1 - z)^2$

c. Let $p = \infty$, $q = 1$, $c = 1$

Then $x^2 (1 - z)^2 - y^2 = (1 - z)^2$

Biquadratic Surface with Elliptical Sections.

Let $f(x'\, y') = x'^2 + y'^2 - c = o$

Then $f\left[\frac{px}{p-z}, \frac{qy}{q-z}\right] = \frac{p^2 x^2}{(p-z)^2} + \frac{q^2 y^2}{(q-z)^2} - c = o$

a. Let $p = 1$, $q = -1$, $c = 1$

Then $x^2 (1 + x)^2 + y^2 (1 - z)^2 = (1 - z^2)^2$

Here the volume between rectilinear directrices is exactly that of a sphere of radius one.

b. Let $p = aq$, $c = 1$

Then $\frac{x^2}{\left(1 - \frac{z}{aq}\right)^2} + \frac{y^2}{\left(1 - \frac{z}{q}\right)^2} = 1.$

Circular sections are at $z = o$ and $z = \frac{2\,aq}{1 + a}$.

The planes $z = o$, $z = q$, $z = \frac{2\,aq}{1 + a}$, $z = aq$ divide every transversal harmonically. In particular every element is divided harmonically by the circular sections and the rectilinear directrices.

c. Combining the last two surfaces and letting $p = aq$,

$$\frac{x^2}{\left(1 - \frac{z}{aq}\right)^2} \pm \frac{y^2}{\left(1 - \frac{z}{q}\right)^2} = C$$

Solve for sections parallel to the xy plane and of the same eccentricity:

$m\left(1 - \frac{z}{aq}\right) \pm \left(1 - \frac{z}{q}\right)$ which gives

$z = \frac{aq\,(m - 1)}{m - a}$ and $z = \frac{aq\,(m + 1)}{m + a}$ for similar conic sections.

It is then easily seen that the four planes,

$z = q$,

$z = \frac{aq\,(m - 1)}{m - a}$

$z = aq$,

$z = \frac{aq\,(m + 1)}{m + a}$,

divide any transversal harmonically.

d. In the most general form with elliptic sections:

Let $p = 1$, $q = \infty$, $c = 1$.

Then $x^2 + (1 - z)^2 y^2 = (1 - z)^2$, the equation of Wallis's Coneo-Cuneus, or the ship carpenter's wedge.

e. Assume case a. The central section at $z = o$ is a circle. Deform the surface by rotating one directrix about the Z axis any angle less than $\pi/2$. The section $z = o$ will now be an ellipse referred to its equi conjugate diameters. The form of the equation of this section will not change; also the form of the equation of the deformed surface will be invariant.

Order of the Resulting Warped Surfaces.

Let $f_n (x\, y)$ represent a homogenous algebraic expression involving x and y and of the nth degree.

In the fundamental demonstration,

1. Let $f\,(x'\, y') = f_1\,(x\, y) - c = o$.

   If x and y are both present, the corresponding warped surface is of the 2d order.

   If x or y is absent, the resulting surface is a plane.

2. Let $f\,(x'\, y') = f_2\,(x\, y) + f_1\,(x\, y) - c = o$.

   x^2 and y^2 both present, 4th order.

   x^2 or y^2 absent, other terms present, 3d order.

   x^2 and y^2 both absent, xy present, x and y present or one or both absent, 2d order.

3. Let $f\,(x'\, y') = f_3\,(x\, y) + f_2\,(x\, y) + f_1\,(x\, y) - c = o$.

   x^3 and y^3 both present, 6th order.

   x^3 or y^3 absent, other terms present, 5th order.

   x^3 and terms involving x^2 absent; or, y^3 and terms involving y^2 absent, 4th order,

   x^3 and y^3 both absent, other terms present, 4th order.

   x^3, y^3, and xy^2 and terms involving y^2 absent, other terms present; or, x^3, y^3, and x^2y and terms involving x^2 absent, other terms present, 3d order.

To deduce the general law of order of the resulting scrolls, construct Fig. 2. Within the squares are present all the powers and combinations that can occur in a complete equation in x, y, of the 5th degree. The numbers at the intersections of the lines show the order of the resulting scroll provided at least two terms remain in our original $f\,(x', y') = o$, one of which lies in a square two sides of which converge in the angle in question, or one of the two terms lies in a square bounded above and to the right by one of the lines converging at the angle, the other in a square

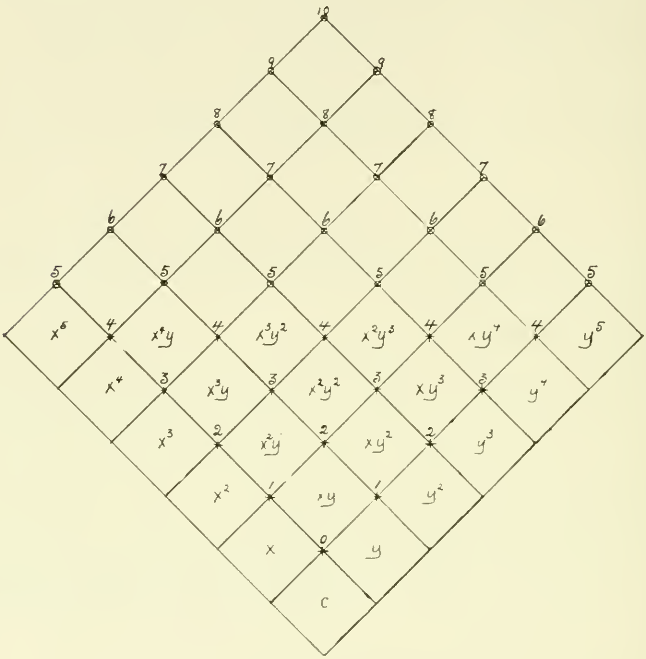

Fig. 2.

bounded above and to the left by the other line making the angle. Thus below one of the points marked 5 is found the term x^3y^2. This term joined with any or all others lying between the lines converging at that particular 5, will yield a scroll of the 5th order.

So also we will have a scroll of the 5th order if we select x^2y^2 on one side and x^3 on the other side of the space bounded by the lines converging at the same point 5.

At the middle point of the whole of Fig. 2 is a vertex marked 4. The following groups can be arranged for the equation of the curvilinear directrix, but in every case the resulting scroll will be of the 4th order.

1. x^2y^2 and c present, xy present or absent,
2. x^2y^2 and c present, and other terms present besides x y,

3. x^2y and xy^2 present, other terms present or absent,

4. x^2y and y^2 present, other terms present or absent (or xy^2 and x^2),

5. x^2 and y^2 present, other terms of lower degree present or absent.

1 and 2 are built from 4th degree terms and the resulting equation is only the 4th.

3, has two 3d degree terms present, scroll 4th.

4, one term 3d degree, other 2d, scroll 4th.

5, built from second degree terms, scroll 4th.

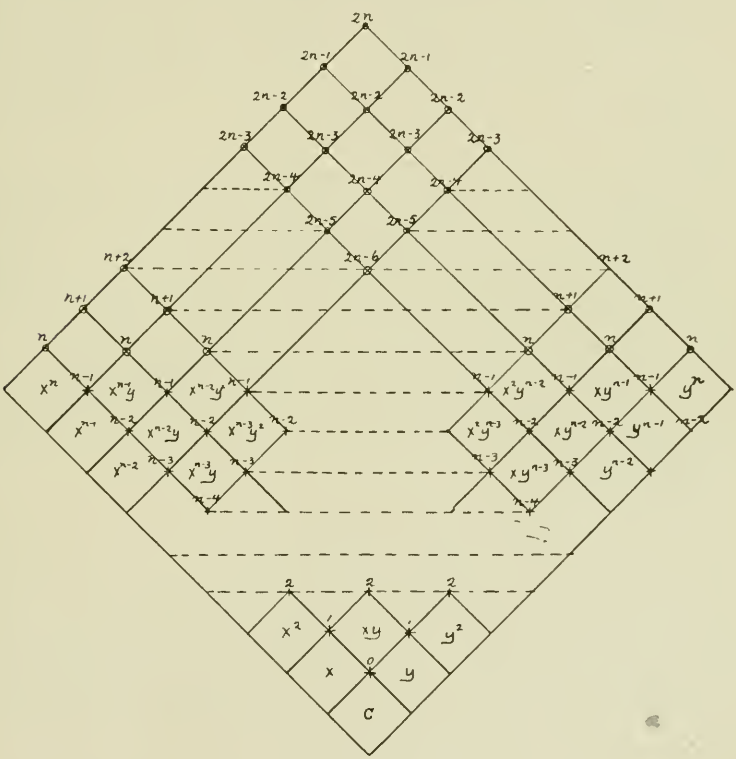

Fig. 3.

Fig. 3, shows at once the order of the resulting scroll when the equation of the curvilinear directrix is marked by the presence or absence of certain specified terms.

Double Generation.

The law of double generation is simply stated. Two straight lines are chosen parallel to the plane of the curvilinear directrix, the three giving rise to a scroll of a certain equation. Suppose two other straight lines can now be found parallel to the plane of the curvilinear directrix and intersecting the first two rectilinear directrices. Suppose the use of the second pair of lines gives exactly the same equation as the first two, then the surface is one of double generation. For example, $x' y' = c$. Substitute $\frac{px}{p-z}$ for x' making $p = 1$ and $\frac{qy}{q-z}$ for y' making $q = -1$. There results $\frac{xy}{(1+z)(1-z)} = c$; now make $p = -1$ and $q = +1$. The same equation results. In fact these are the two generations of the hyperboloid of one sheet.

It then becomes at once apparent that all scrolls are doubly generated whose curvilinear directrix has for its equation a function of the product term (xy), the plane of the curvilinear directrix being parallel to the rectilinear directrices. Thus the first of the five 4th scrolls order mentioned above, viz.: the one having x^2y^2 and c, and perhaps $x\,y$ terms in the equation of the curvilinear directrix is a scoll of double generation.

It is not at once evident that the property discussed above is coextensive with all the doubly generated warped surfaces in the family under discussion. Such surfaces may also depend upon other properties not yet discovered.

General Observations.

It is evident that the validity of the demonstration does not require the axis of Z to be the common perpendicular between the two rectilinear directrices. If the Z axis connects the two directrices in question and passes through the middle point of their common perpendicular, it follows at once that the demonstration proceeds as before by parallel instead of orthogonal projection.

If we conceive the three axes of reference, under the restrictions just given, to be oblique to each other, we find the resulting equations are still in their simplest forms. In the surfaces of the second order the axes would then be conjugate axes. In surfaces of higher order the axes of reference would play the part of conjugate axes.

It will frequently happen that the equation of a scroll will be sought whose three directrices are given as above, viz., two rectilinear and one plane curvilinear directrix, but the latter in some plane not parallel to the two former lines.

In this case additional means should be given for writing the equation of the surface under the new conditions. It will then be easy to find a section parallel to the two right-line directrices and the problem then is solved by the process discussed in this paper.

A modification of the method here discussed finds the equation of a scroll given by two rectilinear directrices and a plane section of the surface, the section being oblique to a plane parallel to the two given straight lines