Newtonian Idea of the Calculus.

Arthur S. Hathaway.

The history of the calculus shows that even a mathematical theory cannot escape the effects of environment. Sir Isaac Newton was for many years the sole possessor of a knowledge of the calculus, and used it with a power which few have been able to equal since his time; yet he has had practically no influence on its present form of development. This was due to Newton's dislike for controversy, so that instead of contending for his ideas, he let them appear only in concise and general form, or even not at all. With the exception of his first two papers on optics, "all of his works were published only after the most persistent solicitations of his friends, and against his own wishes." The criticism which would have aroused an ambitious man to a vigorous defense, had the opposite effect on his disposition. "I was so persecuted," he wrote, "with discussions arising out of my theory of light, that I blamed my own imprudence for parting with so substantial a blessing as my quiet to run after a shadow."

Newton was well versed in the method of fluxions, and the inverse method, that is in differentiation and integration, by the year 1666. In 1669 he circulated a manuscript on the subject among his friends, but refused their solicitations to have it published, and it was not until 1693 that it was communicated to the scientific world by Wallis, in the second volume of his works. During this interval of a quarter of a century, Newton had changed his ideas in important respects, through extensive use of the calculus. He had developed his Theory of Light, discovered the Binomial Theorem, determined the Law of Gravitation, and the Principles of Dynamics, and made important investigations in all departments of mathematical and physical science.

Although the *Principia*, which appeared in 1687, contained no direct information on the calculus, yet its fundamental ideas and principles were involved in every detail of the work. The development of the *Principia* is due to the calculus, but Newton undertook the laborious task of translating everything into the elementary geometrical methods of the time and omitted many results which he had obtained by the calculus, because he could not so interpret them. Many things have been discovered since his time that were afterwards found in his papers and correspondence, and he left many undemonstrated theorems, whose proofs baffled succeeding mathematicians for 50, 100, and even 200 years.

The *Quadrature of Curves*, published in 1704, and the *Principia*, are the proper sources for Newton's matured ideas on the calculus, and not his earlier manuscript, published by Wallis. The earlier paper adopts the infinitesimal method of neglecting small quantities which is now associated with Leibnitz's calculus, not, however, with the latter's disregard of logic, but in connection with the idea of a limit which is the modern foundation of that method.

Newton states in the *Quadrature of Curves* that "in mathematics the minutest errors are not to be neglected." Also,

"I consider mathematical quantities in this place, not as consisting of very small parts, but as described by continuous motion. Lines are described and thereby generated, not by the apposition of parts, but by the continued motion of points; superficies by the motion of lines; solids by the motion of superficies; angles by the rotation of sides; portions of time by continual flux; and so on in other quantities. These geneses really take place in the nature of things and are daily seen in the motion of bodies."

He then goes on to define fluxions, or as we would now call them, differentials:

"Fluxions are as near as we please, as the increments of fluents, generated in times which are the same and as small as possible, and to speak accurately, they are in the prime ratio of nascent increments; yet they can be expressed by any lines whatever which are proportional to them."

Newton immediately illustrates this definition by the abscissa and ordinate of a curve, whose differentials are shown to be any corresponding increments of abscissa and ordinate along the tangent line. This, and numerous similar illustrations in the *Principia*, show that Newton meant by the ultimate ratio of vanishing quantities, *the limit of the ratio of any finite proportionals to the vanishing quantities*. See, for example, Princ. Bk. 1, Lemma 1, Art. 12, "Ultimate Ratio of Vanishing Quantities." Also, Lemmas 7, 8, 9. Newton did not consider the modern question as to whether or not this ratio was definite, and the answer to that question *is not pertinent to his definition*. In other words, differentials can exist when such ratio is indeterminate. Translated into its exact modern equivalent, his definition is:

Corresponding differentials are, as near as we please, proportionals to corresponding and indefinitely small increments of variables, and to speak accurately, they are corresponding limits of such proportionals.

The power and generality of this definition can only be understood after a careful study of its consequences. It applies whatever the number of independent variables. It is the mathematical foundation of Newton's conception of the state of change of variables, in which corresponding differentials are made to signify corresponding increments. In other words, corresponding increments of *a state of change of variables* are as near as we please, proportionals to corresponding and indefinitely small increments of the variables.

As an illustration of the method, consider $z = xy$, and as usual, let $\triangle x$, $\triangle y$, $\triangle z$, denote any corresponding increments of x, y, z. Then, $\triangle z = x\triangle y + y\triangle x + \triangle x \,.\, \triangle y$

Let N be a variable number which becomes indefinitely large in any way whatever (as $N = 1, 2, 3, 4$, and so on indefinitely). Conceive $\triangle x$, $\triangle y$, to diminish as N increases, so that their proportionals, $N\triangle x$, $N\triangle y$, remain finite and approach limits designated by dx, dy ($\triangle x = dx/N + 8/N^2$, $\triangle y = dy/N + 5/N^2$, for example). Then if dz denote the limit of the remaining proportional $N\triangle z$, the equation from which it is to be determined is $N\triangle z = xN\triangle y + yN\triangle x + N\triangle x \,.\, \triangle y$, which gives, by the theorems of limit, $dz = xdy + ydx$.

Here, the ratio dz/dx is absolutely indeterminate, since it depends upon the values chosen for dx, dy.

Leibnitz rediscovered the calculus in 1676, and immediately published his methods and spread them over Europe. His right to the title of independent discoverer was disputed by the friends of Newton, because when Leibnitz was just turning his attention to mathematics in 1673, he visited London and consulted some manuscripts of Newton. Leibnitz's defense is that he did not see the manuscript on the calculus, and his notes taken at the time, and afterwards discovered, contain only references to Newton's papers on optics. It is fortunate in respect to notation that we have received the calculus from the hands of Leibnitz rather than Newton; but the history of the calculus, from Leibnitz on, revolves about objections to his infinitesimal methods. In order to avoid those methods, Lagrange recast the calculus into practically its present form. He regarded the differentials of the independent variables as their small actual increments, and the differential of a dependent variable as that part of its increment which is of first degree when it is expanded in ascending powers of the independent increments. In his method, the principle quantities were the differential coefficients, and if z were a function of x, y, he wrote

$$dz = \frac{dz}{dx} dx + \frac{dz}{dy} dy$$

where dz/dx was a whole symbol for the coefficient of dx in dz, and not the quotient of dz by dx; and similarly for dz/dy.

This idea was not received with favor, partly because it made the calculus depend upon expansions in series, whereas, one important feature of the calculus was the determination of such expansions.

At present, we have a derivative calculus, with a differential notation, in which differentials have significance only in quotient forms; in fact the derivative is Lagrange's differential coefficient, and the two terms are used interchangeably. The student is taught that the quotient form is an inseparable symbol, but the notation, and the calculus itself, eventually require their separation. The explanations which have been devised for such separation of inseparable symbols are sometimes remarkable. The method of rates is simply to define the derivative dy/dx as the rate at which y is changing, and dy, dx, as any quantities whose ratio is dy/dx. This is not the same as Newton's method, who makes dy the amount which y changes in its state of change when x changes by dx, and thence dy/dx is the change of y per unit change of x. It *does* matter whether we make differentials the prime quantities, and thence deduce the significance of their ratios, or whether we make the ratios the prime quantities, and thence deduce differentials. For, two variables can have differentials, with no ratio that is *definite,* i. e., independent of the values of the differentials themselves.

In a calculus in which the derivative is the prime quantity, the differential notation creates numerous *artificial* difficulties which would be eliminated by a proper derivative notation; but this would limit the scope of the calculus and alter many of its time-honored developments. Nor is it necessary to make a change of notation, because the present notation is made completely significant by Newton's definition.

When we consider the weight that attaches to the name of Newton, it would seem that his views on the calculus were worthy of being considered, even today. When we add that he is the original inventor, and that his fundamental idea of the differential is the very one that is needed to give the differential calculus an intelligent and rigorous mathematical basis, it is certainly time that he came into his own.