Some Properties of Binomial Coefficients.

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§1.

The binominal coefficients of the expansion

$$(x+y)^k = \binom{k}{0}x^k + \binom{k}{1}x^{k-1}y + \binom{k}{2}x^{k-2}y^2 + \ldots + \binom{k}{k}y^k$$

were known to possess a simple recursion formula

$$(1) \qquad \binom{k}{n} + \binom{k}{n+1} = \binom{k+1}{n+1} \qquad k, n = 0, 1, 2, 3 \ldots$$

by means of which Pascal's Triangle*

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	*etc.*
$k = 0$	1					
$k = 1$	1	1				
$k = 2$	1	2	1			
$k = 3$	1	3	3	1		
$k = 4$	1	4	6	4	1	
etc.	—	—	—	—	—	—

could be built up, before Newton showed that they are functions of k and n:

$$(2) \qquad \left.\begin{aligned} \binom{k}{n} &= \frac{k(k-1)\ldots(k-n+1)}{n!}, \quad n = 1, 2, 3, \ldots \\ \binom{k}{n} &= 1 \qquad\qquad n = 0 \end{aligned}\right\} k = 0, 1, 2 \ldots$$

A great number of relations involving binomial coefficients have been discovered**; some of the most useful of these are

$$(3) \qquad n\binom{k}{n} = k\binom{k-1}{n-1}; \quad \binom{k}{n} = \frac{k}{k-n}\binom{k-1}{n}; \quad \binom{k}{n} = 0 \text{ if } n > k.$$

*See Chrystal: Algebra I, p. 81.

**See Chrystal: Algebra II, Chaps. XXIII, XXVII. Hagen: Synopsis der hoeheren Mathematik, p. 64: Pascal: Repertorium der hocheren Mathematik I, Kap. II, Sec. 1.

From (1) and (3) it follows that $\binom{k}{n}$ satisfies the linear difference equation

$$(n+1)\,f(n+1) + (n-k)\,f(n) = 0$$

It is well known that the sum of the coefficients $(x+y)^k$ is 2^k and that the sum of the odd numbered coefficients is equal to the sum of the even numbered ones; the following are perhaps not so well known:

(4) If, beginning with the second, the coefficients of $(x-y)^k$ be multiplied by c^n, $(2c)^n$, $(3c)^n$, ... $(kc)^n$ respectively; c being arbitrarily chosen different from zero, the sum of the products will vanish for $n = 1, 2, 3, \ldots\ldots k-1$ but not for $n \geqq k$, e. g.

$k = 8$	$-8,$	$28,$	$-56,$	$70,$	$-56,$	$28,$	$-8,$	1
$c = 2$	$2^n,$	$4^n,$	$6^n,$	$8^n,$	$10^n,$	$12^n,$	$14^n,$	16^n

The sum of the products vanishes for $n = 1, 2 \ldots\ldots 7$; but not for $n > 7$; for $n = 8$ it is 10,321,920.

(5) If the first k coefficients of $(x-y)^{k+1}$ be multiplied term by term, with k^n, $(k-1)^n$, $(k-2)^n$, ... 1^n, (n, $k = 1, 2, 3, \ldots\ldots$) the sum of the products will be

$$(-1)^{k+n} \text{ if } n \leqq k \qquad \text{and } (k+1)!-1 \qquad \text{if } n = k+1;$$

in particular

$$k^k \binom{k+1}{0} - (k-1)^k \binom{k+1}{1} + (k-2)^k \binom{k+2}{2} - \ldots + (-1)^{k-1} \binom{k+1}{k-1} = 1$$

e. g. take $k = 5$.

$1,$	$-6,$	$15,$	$-20,$	$15,$
$5^n,$	$4^n,$	$3^n,$	$2^n,$	$1^n,$

The sum of the products is $+1$, -1, $+1$, -1, $+1$, 719, for $n = 1, 2, 3, 4, 5, 6$, respectively.

Both (4) and (5) are special cases of

(6) If the coefficients of $(x-y)^k$, ($k = 1, 2, 3, \ldots$) be multiplied term by term by the nth powers ($n = 0, 1, 2, \ldots$) of the terms of any arithmetic progression with common difference $d \neq 0$, the sum of the products will vanish if $n<k$; will be $(-d)^k\,(k!)$ if $n = k$; and if $n = k+1$ will be the product of this last result and the sum of the terms of the arithmetic progression.

. g. take $k = 6$, $d = -1$, a.p., 4, 3, 2, etc.

$1,$	$-6,$	$15,$	$-20,$	$15,$	$-6,$	1
$4^n,$	$3^n,$	$2^n,$	$1^n,$	$0^n,$	$(-1)^n$	$(-2)^n$

The sum of the products vanishes for $n = 1, 2, 3, 4, 5$, but not for $n>5$; for $n = 6$, it is $(-1)^6\,(6!) = 720$; and for $n = 7$, it is $720(4 + 3 + 2 + 1 + 0 - 1 - 2) = 5040$.

The third conclusion of (6) shows that if

(I) $a \quad + \quad (a + d) \quad + \quad (a + 2d) \quad + \ldots\ldots\ldots + \quad (a + kd)$

and

(II) $\binom{k}{0} a^k - \binom{k}{1} (a + d)^k + \binom{k}{2} (a + 2d)^k - \ldots\ldots + (-1)^k \binom{k}{k} (a + kd)^k$

be multiplied term by term and the $(k + 1)$ products be added, the result will be the same as though (II) be multiplied through by the terms of (I) in succession and the $(k + 1)^2$ products be added; e.g. take $k = 4$, $a = 1$, $d = 2$

(I)	1 ,	3 ,	5 ,	7 ,	9
(II)	$1 \cdot 1^4$,	$-4 \cdot 3^4$,	$6 \cdot 5^4$,	$-4 \cdot 7^4$,	$1 \cdot 9^4$
	1	— 972	+ 18750	— 67228	+ 59049 = 9600

	$1 \cdot 1^4$	$-4 \cdot 3^4$	$6 \cdot 5^4$	$-4 \cdot 7^4$	$1 \cdot 9^4$	
1	1	— 324	3750	— 9604	6561	384
3	3	— 972	11250	—28812	19683	1152
5	5	—1620	18750	—48020	32805	1920
7	7	—2268	26250	—67228	45927	2688
9	9	—2916	33750	—86436	59049	3456
	25	—8100	+93750	—240100	+164025	9600

§2.

The properties noted above, and many others, can be made to depend upon those of the sum

(1) $$S(k, n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} i^n \qquad k, n = 0, 1, 2, 3, \ldots$$

It is readily shown that

(2) $S(k, n)$ vanishes for $k>n$

(3) $$\left.\begin{aligned} S(k, n) &= -k \sum_{i=k}^{n} \binom{n-1}{i-1} S(k-1, i-1) \\ &= -\sum_{i=k}^{n} \binom{n}{i-1} S(k-1, i-1) \end{aligned}\right\} \quad k, n = 0, 1, 2, 3, \ldots$$

whence $S(k, n)$ is divisible by $k!$ and in fact $S(n, n) = (-1)^n n!$ Also, since $S(1, n) < 0$, it follows that for fixed k, $S(k, n)$ preserves a constant sign (or vanishes) for all values of n: and this sign is the same as that of $(-1)^k$.

These numbers possess a recursion formula

(4) $S(k, n) = k[S(k, n-1) - S(k-1, n-1)]$ $\qquad n, k = 0, 1, 2, \ldots .$

by means of which may be constructed,

A TABLE OF VALUES OF $S(k, n)$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$n=0$	1								
$n=1$	0	—1							
$n=2$	0	—1	2						
$n=3$	0	—1	6	—6					
$n=4$	0	—1	14	—36	24				
$n=5$	0	—1	30	—150	240	—120			
$n=6$	0	—1	62	—540	1560	—1800	720		
$n=7$	0	—1	126	—1806	8400	—16800	15120	—5040	
$n=8$	0	—1	254	—5796	40824	—126000	191520	—141120	40320

Subtract any entry from the one on its right, multiply by the value of k above the latter.

(5) $$\sum_{k=0}^{n} S(k, n) = (-1)^n \qquad \sum_{k=2}^{n} S(k, n) = 1 + \cos n\pi$$

(6) $$\sum_{k=1}^{n} \frac{S(k, n)}{k} = 0 \qquad n = 2, 3, 4, \ldots .$$

(7) $$\sum_{i=k}^{n} \binom{n+1}{i} S(k, i) = (k+1) \sum_{i=k}^{n} \binom{n}{i} S(k, i)$$

(8) $$\sum_{i=k}^{n} \binom{n}{i} S(k, i) = S(k, n) - S(k+1, n)$$

Setting $n = k + 1$ in (7) we obtain

(9) $$S(k, k+1) = \binom{k+1}{2} S(k, k)$$

and similarly we can express $S(k, k+2)$, $S(k, k+3)$, etc. in terms of $S(k, k)$.

From (4)

$$S(k, n) = S(k+1, n) - \frac{1}{k+1} S(k+1, n+1) \qquad k, n = 0, 1, 2, 3, \ldots$$

By applying this m times, we obtain

$$(10) \qquad S(k, n) = \sum_{i=0}^{m} (-1)^i H_i \ S(k+m, n+i)$$

$$k, n = 0, 1, 2, \ldots\ldots ; \ m = 1, 2, 3, \ldots\ldots$$

where H_i is the sum of the products of the fractions $1/(k+1)$, $1/(k+2)$, $1/(k+3)$, ... $1/(k+m)$, taken i at a time; $H_0 = 1$.

The proof of (6) §1 is as follows. If the first term of the arithmetic progression is zero,

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (di)^n = d^n S(k, n)$$

and this vanishes if $n<k$; is $(-d)^k(k!)$ if $n = k$; and is

$$(-d)^k (k!)[d + 2d + 3d + \ldots\ldots + kd] \quad \text{if } n = k+1.$$

If the first term of the arithmetic progression is $a \neq 0$,

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (a+di)^n = d^n \sum_{i=0}^{k} (-1)^i \binom{k}{i} (x+i)^n$$

where $x = a/d \neq 0$.

If we use the notation

$$f(n, x, k) \equiv \sum_{i=0}^{k} (-1)^i \binom{k}{i} (x+i)^n$$

expand $(x+i)^n$ by the binomial formula and reverse the order of summation, we obtain

$$(11) \qquad f(n, x, k) = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} S(k, i)$$

Therefore

$$\begin{aligned} f(n, x, k,) &= 0 \quad \text{when } n<k, \text{ since all the summands vanish} \\ &= S(k, k) \quad \text{when } n = k \\ &= \sum_{i=k}^{n} \binom{n}{i} x^{n-i} S(k, i) \quad \text{when } n>k \end{aligned}$$

In particular, when $n = k+1$

$$f(k+1, x, k) = (x + \frac{k}{2})(k+1)S(k, k) \quad \text{and on putting a/d for } x,$$

$$d^{k+1} f(k+1, x, k) = d^k S(k, k)[a + (a+d) + (a+2d) + \ldots\ldots + (a+kd)]$$

and from these follow the three conclusions* of (6) §1.

*Chrystal: Algebra II, Sec. 9, p. 183, gives the proof of a slightly less general theorem.

Cauchy: Exercices de mathematiques, 1826, I, p. 49 (23), obtains as a by-product the second conclusion of the theorem for the case $d = -1$, and remarks that it is well known.

§3.

In finding the sum of certain series by the method of differences** it is convenient to express positive integral powers of x in terms of the polynomials

(1) $x^{(n)} = x(x-1)(x-2) \ldots\ldots (x-n+1) \qquad n = 1, 2, 3, \ldots.$
$x^{(0)} = 1$

If we set

(2) $x^n = A(o, n)x^{(0)} + A(1, n)x^{(1)} + \ldots\ldots + A(k, n)x^{(k)} + \ldots\ldots + A(n,n)\, x^{(n)}$

it is easily shown that

(3) $$A(k, n) = S(k, n)/S(k, k)$$

whence

(4) $A(k, n)$, $k, n = 0, 1, 2, 3, \ldots$, vanishes if $n<k$; is always positive if $n \geqq k > 0$; in particular $A(n, n) = 1$; and the following relations come from those given in §2 for $S(k, n)$:

(5) $$A(k, n) = \sum_{i=k}^{n} \binom{n-1}{i-1} A(k-1, i-1) = \frac{1}{k} \sum_{i=k}^{n} \binom{n}{i-1} A(k-1, i-1)$$

The recursion formula

(6) $$A(k, n) = k\, A(k, n-1) + A(k-1, n-1)$$

by which may be constructed

A TABLE OF VALUES OF $A(k, n)$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$n = 0$	1								
$n = 1$	0	1							
$n = 2$	0	1	1						
$n = 3$	0	1	3	1					
$n = 4$	0	1	7	6	1				
$n = 5$	0	1	15	25	10	1			
$n = 6$	0	1	31	90	65	15	1		
$n = 7$	0	1	63	301	350	140	21	1	
$n = 8$	0	1	127	966	1701	1050	266	28	1

To any entry add the product of the one on its right and the value of k above the latter.

**See for example Boole's Finite Differences, Chap. IV.

(7) $\sum_{i=k}^{n} \binom{n}{i} A(k, i) = A(k+1, n+1)$ $\qquad n>k = 0, 1, 2, \ldots$

(8) $\sum_{k=1}^{n} A(k, n)\, S(k-1, k-1) = 0$ $\qquad n = 2, 3, 4, \ldots$

Inversely, since

$$x^{(n+1)} = x(x-1)(x-2)\ldots\ldots(x-n) \qquad n = 0, 1, 2, \ldots\ldots$$

if we set

(9) $x^{(n+1(} = x[B(o, n)x^{n} - B(1, n)x^{n-1} + \ldots\ldots + (-1)^{k} B(k, n)x^{n-k} + \ldots\ldots$
$\ldots\ldots + (-1)^{n} B(n, n)]$

it is evident that $B(o, n) = 1$, $n = 0, 1, 2, \ldots\ldots$, $B(k, n)$ = the sum of the products of the numbers 1, 2, 3, ... n, taken k at a time; in particular $B(k, k) = k! = (-1)^{k} S(k, k)$ and $B(k, n) = 0$ if $k>n$. For convenience define $B(p, n) = 0$, if p is a negative integer.

If we multiply both sides of

$$x^{(n)} = x[B(o, n-1)x^{n-1} - B(1, n-1)x^{n-2} + \ldots + (-1)^{n-1}B(n-1, n-1)]$$

by $x - n$, and equate the coefficients of x^{n-k}, we obtain the recursion formula

(10) $$B(k, n) = B(k, n-1) + n\, B(k-1, n-1)$$

by means of which may be constructed

A TABLE OF VALUES OF $B(k, n)$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$n = 0$	1								
$n = 1$	1	1							
$n = 2$	1	3	2						
$n = 3$	1	6	11	6					
$n = 4$	1	10	35	50	24				
$n = 5$	1	15	85	225	274	120			
$n = 6$	1	21	175	735	1624	1764	720		
$n = 7$	1	28	322	1960	6769	13132	13068	5040	
$n = 8$	1	36	546	4536	22449	67284	118124	109584	40320

Multiply any entry by the number $(n+1)$ of the next row, and add to the entry on its right.

(11) $B(k,k+n) = \sum_{i=k}^{n+k} \binom{i}{k} B(k+n-i, k+n-1)$ $\qquad k, n = 0, 1, 2, 3. \ldots$

The equation

$$B(0, n)\, x^n - B(1, n)\, x^{n-1} + \ldots\ldots + (-1)^n B(n, n) = 0$$

has 1, 2, 3, ... n, for roots. If we set

$$S_k = 1^k + 2^k + 3^k + \ldots\ldots + n^k \qquad k = 1, 2, 3\ldots\ldots$$

and solve Newton's formulae* we obtain

$$B(k,k)\; B(k,n) = \begin{vmatrix} S_1 & 1 & 0 & 0 & \ldots\ldots & 0 \\ S_2 & S_1 & 2 & 0 & \ldots\ldots & 0 \\ S_3 & S_2 & S_1 & 3 & \ldots\ldots & 0 \\ S_4 & S_3 & S_2 & S_1 & \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots\ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots\ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots\ldots & \\ S_k & S_{k-1} & S_{k-2} & S_{k-3} & \ldots\ldots & S_1 \end{vmatrix} \quad k,n = 1, 2, 3\ldots$$

This determinant vanishes when $k > n$.

Inversely,

$$S_k = \begin{vmatrix} B(1,n) & B(0,n) & 0 & \ldots\ldots & 0 \\ 2B(2,n) & B(1,n) & B(0,n) & \ldots\ldots & 0 \\ 3B(3,n) & B(2,n) & B(1,n) & \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots\ldots & \\ \ldots & \ldots & \ldots & \ldots\ldots & \\ \ldots & \ldots & \ldots & \ldots\ldots & \\ kB(k,n) & B(k-1,n) & B(k-2,n) & \ldots\ldots & B(1,n) \end{vmatrix}$$

$$k, n = 1, 2, 3\ldots\ldots (\text{even if } k > n)$$

These sums of the powers of the first n natural numbers are connected by the following relations, in which $I(k/2)$ signifies the integral part of $k/2$:

$$\sum_{i=0}^{I(k/2)} \binom{k}{2i+1} S_{2k-1-2i} = \dagger 2^{k-1} S_1^{\,k}$$

$$\sum_{i=0}^{I(k/2)} \frac{2k+1-2i}{1+2i} \binom{k}{2i} S_{2k-2i} = (2n+1)\, 2^{k-1} S_1^{\,k}$$

whence

$$\sum_{i=0}^{k} c_i \binom{k}{i} S_{2k-i} = 0 \qquad \text{where } c_i = \frac{2k+1-i}{1+i} \text{ when } i \text{ is even}$$

$$= -(2n+1) \quad \text{when } i \text{ is odd}$$

*See, for example, Cajori's Theory of Equations, pp. 85–86.

†Stern, Crelle's Journal, Vol. 84, pp. 216–218.

Also

$$\sum_{i=0}^{k} \binom{k+1}{i} S_i = ^{**} (n+1)^{k+1} - 1$$

Relations between the A's and the B's:

$$x^m = \sum_{i=1}^{m} A(i, m)\, x^{(i)} \qquad m = 1, 2, 3 \ldots\ldots$$

$$x^{(i)} = \sum_{j=0}^{i-1} (-1)^j B(j, i-1) x^{i-j} \qquad i = 1, 2, 3, \ldots\ldots$$

Therefore

$$x^m = \sum_{i=1}^{m} A(i, m) \sum_{j=0}^{i-1} (-1)^j B(j, i-1)\, x^{i-j}$$

the coefficient of x^k on the right is

$$\sum_{i=0}^{m-k} (-1)^i A(k+i, m)\ B(i, k+i-1)$$

and this must vanish $k = 1, 2, 3, \ldots\ldots m-1$, and be equal to 1, for $k = m$.

Whence, setting n for $m - k$,

$$\sum_{i=0}^{n} (-1)^i A(k+i, k+n)\, B(i, k+i-1) = 0, \qquad \begin{cases} k = 0, 1, 2, \ldots \\ n = 1, 2, 3, \ldots \end{cases}$$

or, setting i for $k + i$, and n for m,

$$(12) \quad \sum_{i=k}^{n} (-1)^i A(i,n)\, B(i-k, i-1) = 0. \qquad \begin{cases} k = 0, 1, 2, \ldots\ldots\ldots n-1 \\ n = 1, 2, 3, \ldots\ldots\ldots \end{cases}$$

Similarly, starting from

$$x^{(m)} = \sum_{i=0}^{m-1} (-1)^i B(i, m-1)\, x^{m-i}$$

we obtain

$$(13) \quad \sum_{i=0}^{n} (-1)^i A(k, k+n-i)\, B(i, k+n-1) = 0, \qquad \begin{cases} k = 0, 1, 2, \ldots \\ n = 1, 2, 3, \ldots \end{cases}$$

This relation may be generalized as follows:

Set

$$C(k,n,p) = \sum_{i=0}^{n} (-1)^i A(k, k+n-i)\, B(i, k+n-p)$$

**Prestet, Elements de Mathematique, p. 178.

then directly and by (13)

$$\text{(a)}\qquad \left.\begin{array}{ll} C(k,o,p) = 1 & p = 0, 1, 2, \ldots\ldots \\ C(k,n,1) = 0 & n = 1, 2, 3, \ldots\ldots \end{array}\right\} k = 0, 1, 2, \ldots\ldots$$

making use of (10) we obtain

$$\text{(b)}\qquad C(k,n,p) = C(k,n,p-1) + (k+n-p-1)\, C(k,n-1,p-1)$$

The left side vanishes when $p = 1$; therefore

$$C(k,n,0) = -(k+n)\, C(k,n-1,0)$$

By repeating this $(n-1)$ times and noting that $C(k,0,0) = 1$, we obtain

$$\text{(c)}\qquad C(k,n,0) = (-1)^n (k+1)\,(k+2) \ldots . (k+n) \qquad \begin{cases} k = 0, 1, 2, \ldots \\ n = 1, 2, 3, \ldots \end{cases}$$

Setting $p = 2, 3, 4, \ldots\ldots \bar{n}$, in (b), we find

$$\text{(d)}\qquad \begin{array}{ll} C(k,n,p) = 0 & \text{for } p = 1, 2, 3, \ldots\ldots n \\ \qquad\quad\ \ = k^n & \text{when } p = n+1 \end{array}$$

Therefore for all values of $k = 0, 1, 2, \ldots\ldots$; and $n = 1, 2, 3, \ldots\ldots$

$$\text{(14)}\quad \sum_{i=0}^{n} (-1)^i A(k, k+n-i)\, B(i, k+n-p) = (-1)^n (k+1)\,(k+2) \ldots . (k+n) \quad \text{when } p = 0$$

$$= 0 \text{ when } p = 1, 2, 3 \ldots . n$$
$$= k^n \text{ when } p = n+1$$

Example illustrating (14) for $k = 2$, $n = 3$.

		$i=0$	$i=1$	$i=2$	$i=3$	
	$A(2,5-i)$	15	-7	3	-1	sums of products
$p = 0$	$B(i,5)$	1	15	85	225	$(-1)^3\, 3 \cdot 4 \cdot 5$
$p = 1$	$B(i,4)$	1	10	35	50	0
$p = 2$	$B(i,3)$	1	6	11	6	0
$p = 3$	$B(i,2)$	1	3	2	0	0
$p = 4$	$B(i,1)$	1	1	0	0	2^3

In particular, when $p = n$,

$$\sum_{i=0}^{n} (-1)^i A(k, k+n-i)\, B(i, k) = 0$$

or, setting $n-k$ for n

$$(15)\qquad \left.\begin{aligned} &\sum_{i=0}^{n-k} (-1)^i A(k, n-i)\, B(i, k) = 0 \\ &\sum_{i=0}^{k} (-1)^i A(k, n-i)\, B(i, k) = 0 \end{aligned}\right\} \text{ provided } n>k = 0, 1, 2, 3. \ldots$$

The two sums are equivalent since for $i>k$, $B(i, k)$ vanishes and for $i>n-k$, $A(k, n-i)$ vanishes.

From (15)

$$A(k,n) = \sum_{i=1}^{k} (-1)^{1+i} A(k, n-i)\, B(i, k), \qquad n>k = 0, 1, 2, \ldots$$

whence

$$B(k,n) = \sum_{i=1}^{k} (-1)^{1+i} B(k-i, n)\, A(n, n+i), \quad n>k = 0, 1, 2, \ldots$$

Solving for the successive A's and B's, and for brevity writing A_1, A_2 for $A(n, n+1)$, $A(n, n+2)$ etc., and B_1, B_2, for $B(1,k)$, $B(2, k)$ etc.,

$$\begin{aligned} A(k,k) &= 1 \\ A(k,k+1) &= B_1 \\ A(k,k+2) &= B_1^2 - B_2 \\ A(k,k+3) &= B_1^3 - 2B_1B_2 + B_3 \\ A(k,k+4) &= B_1^4 - 3B_1^2B_2 + 2B_1B_3 - B_4 + B_2^2 \\ A(k,k+5) &= B_1^5 - 4B_1^3B_2 + 3B_1^2B_3 - 2B_1B_4 + B_5 + 3B_1B_2^2 - 2B_2B_3 \end{aligned}$$

etc., etc.

$$\begin{aligned} B(0,n) &= 1 \\ B(1,n) &= A_1 \\ B(2,n) &= A_1^2 - A_2 \\ B(3,n) &= A_1^3 - 2A_1A_2 + A_3 \end{aligned}$$

etc., etc., in exactly the same form as the B's.

$S(k,n)$ satisfies the linear difference equation of order k,

$$(16)\quad S(k,n+k) - B(1,k)\, S(k,n+k-1) + \ldots + (-1)^i B(i,k)\, S(k,n+k-i) + \ldots \\ \ldots + (-1)^k B(k,k)\, S(k,n) = 0$$

of which the characteristic equation has for roots $1, 2, 3 \ldots . k$; and the conditions

$$S(k, n) = 0;\ n = 1, 2, 3 \ldots k-1;\ S(k, k) = (-1)^k k!$$

are exactly sufficient to determine the constants. These two equations, therefore, completely characterize

$$S(k,n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} i^n$$

In like manner, the difference equation

$$(17) \quad A(k,n+k) - B(1,k)\,A(k,n+k-1) + \ldots\ldots + (-1)^i B(i,k)\,A(k,n+k-i) + \ldots\ldots + (-1)^k B(k,k)\,A(k,n) = 0$$

and the conditions

$$A(k,n) = 0, \quad n = 1, 2, 3 \ldots\ldots k-1; \; A(k,k) = 1$$

completely characterize $A(k,n) = \dfrac{1}{S(k,k)} \displaystyle\sum_{i=0}^{k} (-1)^i \binom{k}{i} i^n$

$B(k,n)$ satisfies the difference equation of order $2k+1$,

$$(18) \quad B(k, n+2k+1) - \binom{2k+1}{1} B(k, n+2k) + \ldots\ldots + (-1)^i \binom{2k+1}{i} B(k, n+2k+1-i) + \ldots\ldots - B(k,n) = 0$$

of which the characteristic equation is

$$(x-1)^{2k+1} = 0$$

Whence $B(k,n)$ is a polynomial of degree $2k$ in n, but the $k+1$ obvious conditions

$$B(k,n) = 0, \quad n = 0, 1, 2, 3, \ldots\ldots k-1, \quad B(k,k) = k!$$

are not sufficient to determine the constants. It is possible, however, by the successive application of the method of differences, since

$$\triangle B(k,n) = (n+1)\,B(k-1,n)$$

to determine these constants for any particular value of k.

Thus:

$$B(1,n) = \tfrac{1}{2}(n+1)n$$

$$B(2,n) = \frac{1}{24}(n+1)n(n-1)\,(3n+2)$$

$$B(3,n) = \frac{1}{48}(n+1)^2 n^2 (n-1)\,(n-2)$$

etc., etc.

§4.

The properties of

$$f(n,x,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (x+i)^n \qquad §2$$

and an application of $\sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{1}{x+i}$ in the theory of gamma functions suggests the generalization:

$$(1) \qquad f(t,x,k,n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} i^n (x+i)^t$$

$$k,n = 0, 1, 2, 3 \ldots\ldots ; \quad t = 0, \pm 1, \pm 2, \ldots\ldots$$

Whence

$$(2) \quad f(0,x,k,n) = S(k,n) \qquad k,n = 0, 1, 2, \ldots\ldots$$

$$(3) \quad f(t,x,0,n) = x^t \qquad \text{when } n = 0$$

$$= 0 \qquad \text{when } n > 0$$

$$(4) \quad f(t,x,1,n) = x^t - (x+1)^t \qquad \text{when } n = 0$$

$$= -(x+1)^t \qquad \text{when } n > 0$$

When $t < 0$, this function has poles at $x = -1, -2, \ldots\ldots -k$, and also when $n + t < 0$, at $x = 0$.

Since $f(t,x,k,n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} i^n (x+i)^{t-m} (x+i)^m$

we have the recursion formula

$$(5) \qquad f(t,x,k,n) = \sum_{i=0}^{m} \binom{m}{i} x^i f(t-m,x,k,m+n-i)$$

$$t = o, \pm 1, \pm 2, \ldots ; \quad k,n = 0, 1, 2, 3, \ldots ; \quad m = 1, 2, 3, \ldots$$

If t is not negative, we have on setting t for m in (5)

$$(6) \quad f(t,x,k,n) = \sum_{i=0}^{t} \binom{t}{i} x^i S(k,t+n-i) \qquad k,n,t = 0, 1, 2, 3 \ldots$$

If $0 < n \leqq k$

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} i^{(n)} (x+i)^t = (-1)^n k^{(n)} f(t,x+n,k-n, 0) \qquad n = 1, 2, 3 \ldots k$$

Whence, making use of (2) §3,

$$(7) \quad f(t,x,k,n) = \sum_{i=0}^{n} (-1)^i A(i,n)\, k^{(i)} f(t,x+i,k-i,0) \qquad n = 1, 2, 3 \ldots k.$$

In (5), setting $n = 0$, $m = 1$, and $t+1$ for t:

$$f(t+1,x,k,0) = f(t,x,k,1) + x\, f(t,x,k,0)$$

but by (7)

$$f(t,x,k,1) = -k\,f(t,x+1,k-1,0)$$

Therefore,

(8) $x\,f(t,x,k,0) = f(t+1,x,k,0) + k\,f(t,x+1,k-1,0)$, $\qquad k = 1, 2, 3 \ldots$

In (5) setting $t = 0$:

(9) $$\sum_{i=0}^{m} \binom{m}{i} x^i f(-m,x,k,n+m-i) = S(k,n)$$

$$k,n = 0, 1, 2, \ldots\ldots; \qquad m = 1, 2, 3, \ldots\ldots$$

Now $S(k,n)$ vanishes if $k > n$; therefore $f(-m,x,k,n)$ satisfies the linear homo geneous difference equation of order m:

(10) $$\sum_{i=0}^{m} \binom{m}{i} x^i f(-m,x,k,n+m-i) = 0.$$

$$k > n = 0, 1, 2 \ldots \qquad m = 1, 2, 3, \ldots$$

of which the characteristic equation is

$$(r + x)^m = 0$$

whence the complete solution is

(11) $$f(-m,x,k,n) = (c_0 + c_1 n + \ldots\ldots + c_{m-1}\, n^{m-1})\,(-x)^n$$

$$m = 1, 2, 3 \ldots\ldots;\ n = 0, 1, 2, \ldots\ldots k-1; \text{ not for } n \overline{\overline{>}} k;$$

however, the equation (10) itself will give $f(-m,x,k,n)$ for

$$n = k,\ k+1,\ \ldots\ldots\ k+m-1.$$

For $m = 1$, we have

$$f(-1,x,k,n) = c_0\,(-x)^n \qquad n = 0, 1, 2, 3, \ldots\ldots k.$$

and setting $n = 0$, we determine

$$c_0 = f(-1,x,k,0).$$

setting $t = -1$ in (8)

$$\begin{aligned} f(-1,x,k,0) &= \frac{1}{x}\left[S(k,0) + k\,f(-1,x+1,k-1,0)\right] \\ &= \frac{1}{x} \text{ when } k = 0 \\ &= \frac{k}{x}\, f(-1,x+1,k-1,0) \qquad k = 1, 2, 3 \ldots \end{aligned}$$

whence by repetition, and noting (3)

$$f(-1,x,k,0) = \frac{k!}{x(x+1)(x+2)\ldots(x+k)}*$$

and

$$f(-1,x,k,n) = \frac{(-x)^n\, k!}{x(x+1)\ldots\ldots\ldots\ldots(x+k)} \qquad n = 0,\ 1,\ 2,\ 3\ \ldots\ k-1$$

therefore, since by (10), $f(-1,x,k,k) = -x\, f(-1,x,k,k-1)$,

$$(12) \qquad f(-1,x,k,n) = \frac{(-x)^n\, k!}{x(x+1)\ldots\ldots(x+k)} \qquad n = 0,\ 1,\ 2,\ 3\ \ldots\ k, \text{ but not } n>k.$$

Example:

$$\begin{aligned} x(x+1)\,(x+2)\,(x+3)\,(x+4) \sum_{i=0}^{4} (-1)^i \binom{4}{i} \frac{i^n}{x+i} &= 24 && \text{when } n = 0\\ &= -24x && n = 1\\ &= 24x^2 && n = 2\\ &= -24x^3 && n = 3\\ &= 24x^4 && n = 4\\ \text{but} \qquad &= 240x^4 + 840x^3 + 1200x^2 + 576x, && n = 5 \end{aligned}$$

To find the value of $f(-1,x,k,n)$ for $n > k$, set $m = 1$ in (9) and multiply through by

$$(x+1)(x+2)\ \ldots\ (x+k)/S(k,k) = \sum_{i=0}^{k} B(i,k)x^{k-i}/S(k,k)$$

and set

$$g(-1,x,k,n) \quad \text{for} \quad f(-1,x,k,n) \sum_{i=0}^{k} B(i,k)x^{k-i}/S(k,k):$$

$$g(-1,x,k,n+1) = A(k,n) \sum_{i=0}^{k} B(i,k)x^{k-i} - xg(-1,x,k,n)$$

$$k,n = 0,\ 1,\ 2,\ \ldots\ldots$$

Setting $n = k$, $k+1$, we verify that

$$(13) \qquad g(-1,x,k,k+n) = \sum_{j=1}^{n} (-1)^{j-1} A(k,k+n-j) \sum_{i=j}^{k} B(i,k)\, x^{k+j-i-1}$$

holds for $n = 1$, $n = 2$; and a complete induction shows, on taking account of (14) §3, $(p = n)$, that it holds for all positive integral values of n. On

*See Chrystal: Algebra II, Ex. 26, p. 20.

changing the order of summation and replacing $g(-1,x,k,n)$ by its value, we have

$$(14) \qquad f(-1,x,k,n) = \frac{\sum_{j=1}^{k} x^j \sum_{i=1}^{j} (-1)^{i-1} B(k-j+i,k)\, S(k,n-i).}{x(x+1)\,(x+2) \,.\,.\,.\,.\,.\,.\,.\,.\, (x+k)}$$

$$n > k = 0,\ 1,\ 2,\ .\,.\,.\,.\,.\,.\,.\,.$$

the numerator being a polynomial arranged according to ascending powers of x; on arranging this in descending powers of x, taking account of (14) §3,

$$(15) \qquad f(-1,x,k,n) = \frac{\sum_{j=0}^{k-1} x^{k-j} \sum_{i=0}^{j} (-1)^i B(j-i,k)\, S(k,n+i)}{x(x+1)\,(x+2) \,.\,.\,.\,.\,.\,.\, (x+k)}$$

$$n > k = 0,\ 1,\ 2,\ 3\ .\,.\,.\,.\,.\,.\,.\,.$$

It is obvious that (14) does not hold for $n \overline{\overline{<}} k$, since in that case $S(k,n-i)$ vanishes, $i = 1, 2, \ldots\ldots n$; on the other hand, noting that $B(k,n)$ and $S(k,n)$ both vanish if $k > n$ and taking account of (15), §3, it results that in the numerator on the right side of (15), when $n \overline{\overline{<}} k$, the coefficient of every power of x vanishes except that of x^n and this turns out to be

$$(-1)^{k-n} B(0,k) S(k,k) = (-1)^n k! \text{ which agrees with (12).}$$

Therefore,

$$(16) \qquad \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{i^n}{x+i} = \frac{\sum_{j=0}^{k-1} x^{k-j} \sum_{i=0}^{j} (-1)^i B(j-i,k)\; S(k,n+i)}{x(x+1)\,(x+2) \,.\,.\,.\,.\,.\,.\,.(x+k)}$$

$$k,n = 1,\ 2,\ 3\ .\,.\,.\,.\,.\,.\,.\,.$$

but for the case where $n \overline{\overline{<}} k$, (12) is simpler.

Setting $m = 2$ in (11)

$$(17) \quad f(-2,x,k,n) = (c_0 + c_1 n)\,(-x)^n \qquad n = 0,\ 1,\ 2,\ .\,.\,.\,.\,.\,.\,.\,.\ k-1.$$

Put $n = 0$, $n = 1$, and determine

$$c_0 = f(-2,x,k,0)$$

$$c_1 = -\frac{1}{x} f(-2,x,k,1) - f(-2,x,k,0), \qquad \text{which by (7)}$$

$$= \frac{k}{x} f(-2,x+1,k-1,0) - f(-2,x,k,0)$$

In (8) set $t = -2$, $k = 1$

$$x\, f(-2,x,1,0) = f(-1,x,1,0) + f(-2,x+1,0,0)$$

whence by (12) and (3)

$$f(-2,x,1,0) = \frac{1}{x^2(x+1)} + \frac{1}{x(x+1)^2}$$

$$= \frac{1!}{x^2(x+1)^2} \sum_{i=0}^{1} (1+i)\, B(1-i,1)\, x^i$$

Again, setting $k = 2$ in (8)

$$f(-2,x,2,0) = \frac{1}{x} f(-2,x,1,0) + \frac{2}{x} f(-2,x+1,1,0)$$

$$= \frac{2!}{x^2(x+1)^2(x+2)^2} \sum_{i=0}^{2} (1+i)\, B(2-i,2)\, x^i$$

Assume

$$(18) \qquad f(-2,x,k,0) = \frac{k!}{x^2(x+1)^2 \,.\,.\,.\,.\,.\, (x+k)^2} \sum_{i=0}^{k} (1+i)\, B(k-i,k)\, x^i$$

and a complete induction, on taking account of (11) §3, shows that this holds for all positive integral values of k.

Therefore:

$$c_0 = \frac{k!}{x^2(x+1)^2 \,.\,.\,.\,.\,.\, (x+k)^2} \sum_{i=0}^{k} (1+i)\, B(k-i,\mathrm{k})\, x^i$$

$$-c_1 = \frac{k!}{x^2(x+1)^2 \,.\,.\,.\,.\,.\, (x+k)^2} \sum_{i=0}^{k} B(k-i,k)\, x^i$$

and

$$(19) \qquad f(-2,x,k,n) = \frac{(-x)^n\, k!}{x^2(x+1)^2 \,.\,.\,.\,.\,.\, (x+k)^2} \sum_{i=0}^{k} (1+i-n)\, B(k-i,k)\, x^i$$

$$k = 0, 1, 2 \ldots;\ n = 1, 2, 3 \,.\,.\,.\,.\,.\,.\, k-1$$

On computing, by means of (10), the values of $f(-2, x, k, k)$ and $f(-2, x, k, k+1)$, we verify that (19) holds for $n = 1, 2, 3 \,.\,.\,.\,.\,.\,.\,.\, k+1$ but not for $n > k+1$,

Therefore,

$$(20) \qquad \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{i^n}{(x+i)^2} = \frac{(-x)^n\, k!}{x^2(x+1)^2 \,.\,.\,.\,.\, (x+k)^2} \sum_{i=0}^{k} (1+i-n)\, B(k-i,k) x^i$$

$$k = 0, 1, 2, \,.\,.\,.\,.\,.\,;\ n = 0, 1, 2, \,.\,.\,.\,.\,.\, k+1;\ \text{not } n > k+1$$

The corresponding results for $n = k+2,\ k+3$, etc., may be found by putting these values successively for n in

$$(21) \qquad f(-2,x,k,n+2) = S(k,n) - 2x\, f(-2,x,k,n+1) - x^2 f(-2,x,k,n)$$

which results from setting $m = 2$ in (9). The general result may be put into the form

$$(22) \quad f(-2,x,k,n) = \frac{\sum_{j=0}^{2k-2} x^{2k-j} \sum_{i=0}^{k-1} D(i,j,k)\; S(k-i,n)}{x^2(x+1)^2 \;.\;.\;.\;.\;.\;.\;.\; (x+k)^2}; \qquad k,n = 1, 2, 3 \;.\;.\;.$$

in which the coefficients D, are independent of n:

$$D(i,0,k) = 1 \quad \text{when } i = 0$$

$$= 0 \qquad i = 1, 2, 3 \;.\;.\;.\;.\;.\;.\;.$$

$$D(0,j,k) = \sum_{t=0}^{j} B(t,\, k-1)\; B(j-t,\, k-1) \qquad j = 1, 2, 3 \;.\;.\;.$$

but I have not been able to determine a general formula for $D(i,j,k)$ by means of which to calculate the coefficients of $f(-2,x,k,p)$, $p>k+1$, without first calculating successively those for $n = k+2,\ k+3,\ .\ .\ .\ .\ .\ p-1$.

By making use of (10) § 2, (21) may be reduced to

$$(23) \quad f(-2,x,k,n) = \frac{\sum_{j=0}^{2k-2} x^{2k-j} \sum_{i=0}^{k-1} E(i,j,k)\; S(k,n+i)}{x^2(x+1)^2 \;.\;.\;.\;.\;.\;.\;.\; (x+k)^2}; \qquad k,n = 1, 2, 3 \;.\;.\;.$$

with which compare (16)

Example:

$$x^2(x+1)^2(x+2)^2(x+3)^2(x+4)^2 \sum_{i=0}^{4} (-1)^i \binom{4}{i} \frac{i^n}{(x+i)^2} = S(4,n)\, x^8 +$$

$$[12\, S(4,n) + 8\, S(3,n)]\, x^7 +$$
$$[58\, S(4,n) + 76\, S(3,n) + 36\, S(2,n)]\, x^5 +$$
$$[144\, S(4,n) + 272\, S(3,n) + 288\, S(2,n) + 96\, S(1,n)]\, x^5 +$$
$$[193\, S(4,n) + 460\, S(3,n) + 780\, S(2,n) + 720\, S(1,n)]\, x^4 +$$
$$[132\, S(4,n) + 368\, S(3,n) + 840\, S(2,n) + 1680\, S(1,n)]\, x^3 +$$
$$[36\, S(4,n) + 112\, S(3,n) + 312\, S(2,n) + 1200\, S(1,n)]\, x^2$$
$$n = 1, 2, 3 \;.\;.\;.\;.\;.\;.$$

also:

$$= S(4,n)\,x^8 + [20\,S(4,n) - 2\,S(4,n+1)]\,x^7 +$$
$$[170\,S(4,n) - 40\,S(4,n+1) + 35\,S(4,n+2)]\,x^6 +$$
$$[800\,S(4,n) - 340\,S(4,n+1) + 60\,S(4,n+2) - 4\,S(4,n+3)]\,x^5 +$$
$$[2153\,S(4,n) - 1350\,S(4,n+1) + 335\,S(4,n+2) - 30\,S(4,n+3)]x^4 +$$
$$[3020\,S(4,n) - 2402\,S(4,n+1) + 700\,S(4,n+2) - 70\,S(4,n+3)]x^3 +$$
$$[1660\,S(4,n) - 1510\,S(4,n+1) + 476\,S(4,n+2) - 50\,S(4,n+3)]x^2$$
$$n = 1, 2, 3 \ldots\ldots$$

These results are consistent with (20) for $n = 1, 2, 3, 4, 5$ and for $n = 6$ give

$$1560\,x^8 + 14400\,x^7 + 51672\,x^6 + 59520\,x^5 + 100320\,x^4 + 57600\,x^3 + 13824\,x^2.$$

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