Concerning Differential Invariants.

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During the last forty years wonderful progress has been made in many fields of higher mathematics. One distinct line of investigation has had to do with a *microscopic* examination of the fundamental axioms of the elementary mathematics, of conditions of convergence, of the sufficient conditions in the calculus of variations, and so on. Another essential advance has been made by unifying many separate and apparently distinct fields of mathematics under one common law. Among many advances in this latter line of work, none are more important than the work of Sophus Lie, a Norwegian, who lived from 1842 to 1899.

Lie received his doctorate from the University of Christiania in 1865, caring no more for mathematical work than for literary or philological work. In fact, he had thought of becoming an engineer; but receiving an appointment to a docentship in the university, he turned his attention to the study of advanced mathematics. The real mathematical genius of Lie was aroused by a course of lectures on substitutions by Professor Sylow. Lie's creative period seems to have extended from 1868 to about 1874, during which time he came into possession of the essential features of his epoch-making Theory of Continuous Groups. The remainder of his life was devoted to the elaboration of his early conceptions, and to the applications of his theories. A general development of the higher number systems, a classification of ordinary and partial differential equations, with methods of their solutions, invariants and covariants, many problems of physics and astronomy, are all treated from the standpoint of the continuous group. Below is sketched a brief outline of the continuous group theory of Lie, as applied to differential invariants, and the calculation of an important differential invariant is indicated.

1. *Point Transformation.* Let x, y be the Cartesian coördinates of any point in the plane, and let x_1, y_1 be any point other than x, y. Then

$$x_1 = \Phi(x, y), \quad y_1 = \Psi(x, y)$$

is said to be a point transformation, carrying point x, y into point x_1, y_1. Here it is assumed that inversely

$$x = \Phi_1 (x_1, y_1), \; y = \Psi_1 (x_1, y_1)$$

carries the point from x_1, y_1 back to x, y. A point transformation may be looked upon either as a transference of axes from one system to another, not necessarlly the same kind of system, or it may be considered as an actual transference of one point into another position in the plane, the axes of reference remaining unchanged.

2. *Group of Transformations.* A point transformation containing one or more parameters

$$x_1 = \Phi (x, y, a, b, c, \ldots k),$$
$$y_1 = \Psi (x, y, a, b, c, \ldots k),$$

such that for a_0, b_0, c_0, ...k_0, the point x, y transforms into itself, is said to constitute a group of transformations when a succession of two such operations may be replaced by one of the same species. That is, if

$$x_2 = \Phi (x_1, y_1, a_1, b_1, c_1 \ldots) = \Phi \left\{ \Phi (x, y, a, \ldots k), \Psi (x, y, a, \ldots k), a_1 \ldots k_1 \right\}$$

$$= \Phi (x, y, a_2, b_2, c_2, \ldots k_2),$$

$$y_2 = \Psi (x, y, a_2, b_2, c_2, \ldots k_2),$$

where $a_2 = f_1 (a, b, \ldots k, a_1, b_1, c, \ldots k_1)$, $b_2 = f_2 (a, b, \ldots k_1)$ ...., then

$$x_1 = \Phi (x, y, a, b, \ldots k), \; y_1 = \Psi (x, y, a, b, \ldots k)$$

are the transformations of an r-parameter group, the parameters a, b, c, ... k being r in number and independent. A similar definition may be given to a group in one, three, four, or n variables*.

3. *The Infinitesimal Transformations.* An infinitesimal transformation is defined analytically by

$$\delta x = \xi (x, y) \, \delta t, \; \delta y = \eta (x, y) \, \delta t.$$

Such a transformation attaches to any point x, y an infinitesimal motion whose projections on the x —, and y — axes are respectively, $\xi \delta t$ and $\eta \delta t$, and whose distance is $\sqrt{\xi^2 + \eta^2} \, \delta t$. Lie shows such infinitesimal transformations to belong to a single-parameter group.

$$x_1 = \Phi (x, y, a), \; y_1 = \Psi (x, y, a).$$

This may be easily seen by letting a_0 be the value of a which leaves x, y fixed; then

$$x_1 = \Phi (x, y, a_0 + \delta a), \; y_1 = \Psi (x, y, a_0 + \delta a)$$

*See Lie—Scheffers, Differential-gleichungen, pp. 24-25.

give to the point x, y an infinitesimal motion. Expanding in powers of δa, we have*

$$x_1 = \Phi(x, y, a_0) + \left\{\frac{d\,\Phi(x, y, a_0)}{d\,a_0}\right\}\delta a + \ldots,$$

$$y_1 = \Psi(x, y, a_0) + \left\{\frac{d\,\Psi(x, y, a_0)}{d\,a_0}\right\}\delta a + \ldots.$$

But $\Phi(x, y, a_0) = x$, $\Psi(x, y, a_0) = y$, hence

$$x_1 = x + \left\{\frac{d\,\Phi}{d\,a_0}\right\}\delta a + \ldots,$$

$$y_1 = y + \left\{\frac{d\,\Psi}{d\,a_0}\right\}\delta a + \ldots,$$

$$\delta x = \left\{\frac{d\,\phi}{d\,a_0}\right\}\delta a + \ldots \qquad \equiv \xi(x, y)\,\delta t + \ldots,$$

$$\delta y = \left\{\frac{d\,\psi}{d\,a_0}\right\}\delta a + \ldots \qquad \equiv \eta(x, y)\,\delta t + \ldots.$$

Omitting infinitesimals of higher order we have the relations

$$\delta x = \xi(x, y)\,\delta t, \quad \delta y = \eta(x, y)\,\delta t$$

as the infinitesimal transformations of a one-parameter group.

In the notation of Lie the symbol

$$U f \equiv \xi(x, y)\left\{\frac{d\,f}{d\,x}\right\} + \eta(x, y)\left\{\frac{d\,f}{d\,y}\right\},$$

denoting the variation which a function $f(x, y)$ undergoes when x, y receive the increments δx, δy, is employed as the symbol of an infinitesimal transformation. Writing p, q instead of the partial derivative of $f(x, y)$ with respect to x and y, respectively, we have

$$U f \equiv \xi(x, y)\,p + \eta(x, y)\,q.$$

The infinitesinal transformations of an r-parameter group would be given by the symbol

$$U_k f \equiv \xi_k(x, y)\,p + \eta_k(x, y)\,q, \quad k = 1, 2, 3, \ldots r.$$

4. *The Group Criterion.* One of Lie's fundamental theorems furnishes a test whether or not any given set of infinitesimal transformations, $U_k f$, $k = 1, 2, \ldots r$, actually forms a group. This test is the application of Jacobi's *bracket expression*

$$U_i(U_j f) - U_j(U_i f), \quad (i, j = 1, 2, \ldots r, \text{ in all combinations}).$$

*In this article the symbol $\left\{\frac{d\ f}{d\ x}\right\}$ will be used to denote the partial derivative of f with regard to x, instead of the round d usually employed.

If the Jacobi bracket-expression, constructed for all combinations of i, j, is equivalent to a linear function of the symbols $U_k f$ with constant coefficients, then are the symbols

$$U_k f \equiv \xi_k (x, y)\, p + \eta_k (x, y)\, q, \quad k = 1, 2, \ldots r,$$

the infinitesimal transformations of an r-parameter group.*

5. *The Extended Group.* An infinitesimal transformation

$$U f = \xi (x, y) \left\{\frac{d f}{d x}\right\} + \eta (x, y) \left\{\frac{d f}{d y}\right\}$$

may be *extended* in two ways. In the first place, the variation of the coör-dinates of n points is simply the sum of the variations of the coördi-nates of the separate points; hence, $U f$ extended in this manner becomes

$$\text{(A).}\quad U f_n \equiv \sum_{k=1}^{k=n} \left\{ \xi_k (x_k, y_k) \left\{\frac{d f}{d x_k}\right\} + \eta (x_k, y_k) \left\{\frac{d f}{d y_k}\right\} \right\}.$$

The symbol $U f$ may also be extended so as to include the variation of $y' = \frac{d y}{d x}$, $y'' = \frac{d^2 y}{d x^2}$, ..., $y^{(n)} = \frac{d^n y}{d x^n}$. We have

$$\delta x = \xi (x, y)\, \delta t, \quad \delta y = \eta (x, y)\, \delta t.$$

$$\delta y' = \delta \frac{dy}{dx} = \frac{dx\, \delta\, dy - dy\, \delta\, dx}{dx^2} = \frac{d\, \delta\, y - y' d \delta\, x}{dx}$$

$$= \left\{ \frac{d\, \eta}{dx} - y' \frac{d\, \xi}{dx} \right\} \delta\, t = \left\{ \eta_x + y' (\eta_y - \xi_x) - y'^2\, \xi_y \right\} \delta\, t$$

$$= \eta' (x, y, y')\, \delta\, t.$$

In a similar manner,

$$\delta y'' = \left\{ \frac{d\, \eta'}{dx} - y'' \frac{d\, \xi}{dx} \right\} \delta\, t = \eta'' (x, y, y', y'')\, \delta\, t,$$

and so on for higher variations.

The infinitesimal transformation $U f$ extended to include these higher variations becomes

$$\text{(B).}\quad U f_n = \xi \left\{\frac{df}{dx}\right\} + \eta \left\{\frac{df}{dy}\right\} + \eta' \left\{\frac{df}{dy'}\right\} + \eta'' \left\{\frac{df}{dy''}\right\} + \ldots + \eta^{(n)} \left\{\frac{df}{dy_n}\right\}$$

Each of the members of an r-parameter group $U_k f$, $k = 1, 2, \ldots r$, may be extended, giving the infinitesimal transformations of the coördinates of n points as indicated by equation (A); or each may be extended as in (B) to include the variations of x, y, y', y'', y''', ... $y^{(n)}$. A group of transformations extended in style of (A) or (B) is called an extended group.

*Lie—Scheffers, Continuierliche Gruppen, p. 390.

6. *Invariant Functions.* The variation of any function Φ (x, y) when operated upon by an infinitesimal transformation.

$$Uf \equiv \xi p + \eta q$$

is given by

$$U\Phi \equiv \xi \left\{\frac{d\Phi}{dx}\right\} + \eta \left\{\frac{d\Phi}{dy}\right\}.$$

If Φ (x, y) is to remain unchanged, then $U\Phi \equiv o$, and Φ (x, y) is a solution of the homogeneous linear partial differential equation

$$Uf \equiv \xi p + \eta q = o,$$

that is, Φ (x, y) is an integral of Lagrange's equation

$$\frac{dx}{\xi} = \frac{dy}{\eta}.$$

Φ (x, y) so determined is called an invariant for the transformation

$$Uf \equiv \xi p + \eta q$$

A group of two or more independent transformations will not in general have an invariant function. But when extended to include the coördinates of *n* points, as in (A) above, an *r*-parameter group

$$U_k f_{(n)} \equiv \sum_{i=1}^{n} \left\{ \xi_k (x_i, y_i) \left\{\frac{df}{dx_i}\right\} + \eta_k (x_i, y_i) \left\{\frac{df}{dy_i}\right\} \right\}, \ k = 1, 2, \ldots r,$$

gives rise to 2 n — r independent functions

$$\phi_1 (x_1, y_1, \ldots x_n, y_n), \phi_2, \phi_3, \ldots \phi_{2n-r},$$

which are *point-invariants* of the group $U_k f$, and which are derived by integrating the *r* partial differential equations $U_1 f_n = o$, $U_2 f_n = o$, ... $U_r f_n = o$.

After the manner here indicated the writer has calculated all the point-invariants for the twenty-seven finite continuous groups of the plane as classified by Lie.* The results appear in the Proceedings of the Indiana Academy of Science, 1898, pp. 119-135.

7. *Differential Invariants.* An infinitesimal transformation extended to include the increment of y′ leaves invariant two functions ϕ_1 (x, y, y′), ϕ_2 (x, y, y′), the solutions of

$$U'f \equiv \xi p + \eta q + \eta' \left\{\frac{df}{dy'}\right\} = o.$$

The functions ϕ_1, ϕ_2 are called differential invariants of the infinitesimal transformation $U'f$. Lie shows that when two independent differential

*See Lie-Scheffers, Contin. Gruppen, pp. 360-362.

invariants of a given transformation are known, then all others may be found by differentiation.*

$$\phi_3 = \frac{d\phi_2}{d\phi_1},\ \phi_4 = \frac{d\phi_3}{d\phi_1},\ \ldots .$$

An r-parameter group $U_k f$ extended to include the increments of $y', y'', \ldots y^{(r)}$, when equated to zero, gives r partial differential equations in $r + 2$ variables. These r equations have two independent solutions, $\phi_1(x, y, y', \ldots y^{(r)})$, $\phi_2(x, y, y', \ldots y^{(r)})$, which are differential invariants of the r-parameter group. After the plan here indicated Lie has calculated the differential invariants for the twenty-seven groups of the plane.

The calculation of differential invariants may be made by an entirely different method than that used by Lie, and indeed without any knowledge of the group extended as indicated above. A knowledge of the form of a point invariant for the group is necessary.

Let a point invariant $\phi(x_1, y_1, x_2, y_2, \ldots)$ be given, and suppose the points $x_1, y_1; x_2, y_2; \ldots; x_n, y_n$, to be located upon a plane curve

$$x = f_1(t),\ y = f_2(t).$$

Then we would have

$$x_1 = f_1(t_1),\ y_1 = f_2(t_1),\ \ldots\ x_n = f_1(t_n),\ y_n = f_2(t_n),$$

Allowing $x_2, y_2; x_3, y_3; \ldots; x_n, y_n$ to coalesce toward $x_1\, y_1$, we may then expand $x_2, y_2, \ldots.$ in power-series

$$\text{(I)}\ \left\{\begin{array}{ll} x_2 = x_1 + x' dt_2 + x'' \dfrac{dt_2^2}{2} + \ldots, & y_2 = y_1 + (y') dt_2 + (y'') \dfrac{dt_2^2}{2} + \ldots, \\ x_3 = x_1 + x' dt_3 + x'' \dfrac{dt_3^2}{2} + \ldots, & y_3 = y_1 + (y') dt_3 - (y'') \dfrac{dt_3^2}{2} + \ldots, \end{array}\right.$$

and so on for $x_4, y_4, \ldots x_n, y_n$, where

$$\text{(1)}\qquad x' = \frac{dx_1}{dt_1},\quad x'' = \frac{d^2x_1}{dt_1^{\,2}},\quad x''' = \frac{d^3x_1}{dt_1^{\,3}},\ \ldots.,$$

$$\text{(2)}\qquad (y') = \frac{dy_1}{dt_1},\quad (y'') = \frac{d^2y_1}{dt_1^2},\quad (y''') = \frac{d^3y_1}{dt_1^3},\ \ldots..$$

The notation of (1), (2) should be changed from parameter notation to the ordinary $y' = \dfrac{dy}{dx}$, $y'' = \dfrac{d^2y}{dx^2}$, ...

$$\text{(3)}\ \left\{\begin{array}{l} y' = \dfrac{dy}{dx} = \dfrac{(y')}{x'},\ \text{hence } (y') = y' x',\ \text{similarly}, \\ (y'') = y''(x')^2 + y' x'';\ (y''') = y'''(x')^3 + 3y'' x' x'' + y' x'''; \\ (y^{iv}) = y^{iv}(x')^4 + 6y'''(x')^2 x'' + 3y''(x'')^2 + 4y'' x' x''' + y' x^{iv}; \\ (y^{v}) = y^{v}(x')^5 + 10 y^{iv}(x')^3 x'' + y''\left(15 x'(x'')^2 + 10(x')^2 x'''\right) + \\ \qquad y''\left(10 x'' x''' + 5 x' x^{iv}\right) + y' x^{v}, \end{array}\right.$$

and so on for higher derivatives.

*Lie, Math. Annalen, Bd. XXXII.

If in any point invariant ϕ, the values of $x_2, y_2; x_3, y_3 \ldots$, taken from (I) be substituted, and then the result developed into in infinite power-series in the ascending powers of $dt_2, dt_3, dt_4, \ldots dt_n$, the successive coefficients of the separate powers of $dt_2, dt_3, \ldots$, and of the products $dt_2, dt_3, \ldots$ are all invariant functions of $x', x'', x''', \ldots, (y'), (y''), (y'''), \ldots$. These separate invariant functions may then be changed by means of equations (3) above so that only $x', x'', x''', x^{\prime v}, \ldots$ and $y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}, \ldots$, occur. Then by algebraic manipulation the parameters $x', x'', x''', \ldots$ may be eliminated, leaving a differential invariant for the continuous group from which the point invariant ϕ had been derived.

8. *The Differential Invariants for the General Projective Group.*

The general projective group: $p, q, xq, xp - yq, yp, xp + yq, x^2p + xyq, xyp + y^2q$, when extended leaves invariant the point-function.

$$Q = \left\{ \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_5 & y_5 & 1 \end{vmatrix} \div \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} \right\} \div \left\{ \begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_5 & y_5 & 1 \end{vmatrix} \div \begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} \right\}.^*$$

Substituting in Q the series expansions of $x_2, y_2, x_3, y_3, \ldots x_5, y_5$ from equations (I), and developing the determinants, we have the ratio of infinite series which may be further developed into a single power series of the form

$$Q^1 = a_0 + a_1, \left\{\frac{I_2}{I_1}\right\} + a_2 \left\{\frac{I_3}{I_1}\right\} + a_3 \left\{\frac{I_4}{I_1}\right\} + \ldots,$$

where a_i is an expression containing a function of dt_2, dt_3, dt_4, dt_5 to degree i, and where

$$\begin{aligned}
I_1 &= x'(y'') - x''(y') = y''x'^3, \\
I_2 &= x'(y''') - x'''(y') = y'''(x')^4 + 3y''(x')^2x'', \\
(K) \quad I_3 &= x'(y^{iv}) - x^{iv}(y') = y^{iv}(x')^5 + 6y'''(x')^3x'' + 3y''x'(x'')^2 \\
&\quad + 4y''(x')^2x''', \\
I_4 &= x''(y''') - x'''(y'') = y'''(x')^3x'' - y''(x')^2x''' + 3y''x'(x'')^2
\end{aligned}$$

and so on until all orders of differentials $y', y'', y''', \ldots, y^{viii}$ have been included. Now the separate ratios $I_2 : I_1, I_3 : I_1, I_4 : I_1, \ldots$, are separately invariant, and when reduced as in equations (K) contain the arbitary parameters $x', x'', x''', \quad x^{viii}$. The elimination of these parameters is

*See Pro. Ind. Acad., 1898, p. 135.

a tedious process, and will not be indicated here. When performed, however, there results the two differential invariants

$$\phi_1 = \left[2\,A_3\,A_5 - 35\,A_2\,A_3^2 - 7\left\{ A_4 - \frac{5}{3} A_2^2 \right\}^2 \right] \div (A_3)^{\frac{8}{3}},$$

$$\phi_2 = \left[A_3\left\{ A_6 - 84\,A_3\,A_4 + \frac{245}{3} A_2^3 \right\} - 12\left\{ A_5 - \frac{35}{2} A_2\,A_3 \right\}\left\{ A_4 - \frac{5}{3} A_2^2 \right\} + \frac{28}{3}\left\{ A_4 - \frac{5}{3} A_2^2 \right\}^2 \right] \div A_3^3,$$

where

$$A_2 = 3y^{\prime v}\,y'' - 4\,(y''')^2,$$

$$A^3 = y^{v}\,(y'')^2 - 15y^{\prime v}\,y'''\,y'' + \frac{40}{3}(y''')^3,$$

$$A_4 = 3y^{v\prime}\,(y'')^3 - 24y^{v}\,y'''\,(y'')^2 + 60y^{\prime v}\,(y''')^2\,y'' - 40\,(y''')^4,$$

$$A_5 = 9y^{v\prime\prime}\,(y'')^4 - 105y^{v\prime}\,(y'')^3\,y''' + 420\,y^{v}\,(y''')^2\,(y'')^2 - 700y^{\prime v}\,(y''')^3\,y'' + \frac{1120}{3}(y''')^5,$$

$$A_6 = 27y^{v\prime\prime\prime}\,(y'')^5 - 48\,A_5\,y''' - 840\,A_4\,(y''')^2 - 2240\,A_3\,(y''')^3 - 2800\,A_2\,(y''')^4 - \frac{2240}{3}(y''')^6.$$