Gamma Coefficients and Series.

I. The Coefficients.

1. The function.

$$(axby\cdot\cdot) = (ax+by+\cdot\cdot)\ \frac{\Gamma(x+y+\cdot\cdot)}{\Gamma(x+1)\,\Gamma(y+1)\cdot\cdot}$$

will be called *a gamma coefficient of coördinates* x, y, $\cdot\cdot$, *and parameters* a, b, $\cdot\cdot$, and *a multinomial coefficient* when each parameter is unity. We shall use Greek letters to denote coördinates taken from the series 0, 1, 2, 3, $\cdot\cdot$.

At points of discontinuity, the sum of the coördinates is *zero* or a *negative integer*. These points are excluded in the following properties.

2. *A gamma coefficient with a negative integral coördinate is zero.*

3. *Zero coördinates and their parameters may be omitted*, as $(axbyc0) = (axby)$.

4. *The gamma coefficient of a point upon an axis equals the parameter of that axis*, as $(ax) = a$.

5. *The gamma coefficient of any point is the sum of the gamma coefficient of the preceding points* (a preceding point being found by diminishing one coördinate by *a unit*). Let $E\eta$ operate to diminish the n'th coördinate by a unit, then in symbols, *(Note)

$$(axby\cdot\cdot) = (E_1+E_2+\,.\,.)(axby\cdot\cdot)$$

This may be extended to the n'th repetition of $E_1+E_2+\cdot\cdot = 1$, where the E's combine by the laws of numbers.

6. The above property furnishes an immediate proof of the multinomial theorem. Thus let

$$Fn = \Sigma(1\alpha1\beta\cdot\cdot)\ p^{\alpha}q^{\beta}\cdot\cdot,\ \alpha+\beta+\cdot\cdot = n$$

i. e. the summation extends to every point the sum of whose coördinates is n, there being a given number of variables p, q, $\cdot\cdot$, and corresponding integral coördinates α, β, $\cdot\cdot$. Applying art. 5 to the coefficients of Fn, we find $Fn = (p+q+\cdot\cdot)F(n-1)$, and since $F1 = p+q+\cdot\cdot$, therefore $Fn = (p+q+\cdot\cdot)^{\eta}$.

7. *Zero parameters and corresponding coördinates may be omitted, if the result be multiplied by the multinomial coefficient of the omitted coördinates and one other, the sum, less* 1, *of the retained coördinates, as,*

$$(0x0ybzcw) = (bzcw)\ (1x1y1w'),\ w' = z+w-1$$

8. *Equal parameters and their coördinates may be omitted, except one to*

* (Note) Read n for η throughout this paper.

a coördinate the sum of the omitted coördinates, if the result be multiplied by the multinomial coefficient of the omitted coördinates, as

$$(axaybz) = (ax'bz)(1x1y),\ x' = x+y.$$

9. *The coefficient of a parameter of a gamma coefficient is the multinoimal coefficient of the corresponding preceding point.* In symbols,

$$(axby\cdot\cdot) = (aE_1+bE_2+\cdot\cdot)(1x+1y\cdot\cdot)$$

II. Gamma Series.

10. Let there be m variables, p_1, p_2, $\cdot\cdot$, of weights 1, 2, $\cdot\cdot$, and m corresponding parameters, a_1, a_2, $\cdot\cdot$. The *gamma series of weight n* is the sum of all terms in the variables of weight n, each multiplied by the gamma coefficient of its exponents and the corresponding parameters:

$$(a)\quad (ap)n = \Sigma(a_1\alpha_1 a_2\alpha_2\cdot\cdot)\ p_1^{\alpha_1}p_2^{\alpha_2}\cdot\cdot,\ \alpha_1+2\alpha_2+\cdot\cdot = n.$$

This series is not a function of an r'th variable and parameter for $r>n$, since the simultaneous exponent and coördinate αr, is zero.

By applying art. 5 to the coefficients of $(ap)n$, we have,

$$\text{(b)}\ (ap)n = p_1(ap)(n-1)+..+p_{\eta-1}(ap)1+a_\eta p_\eta$$

where, if $r>m$, $p_r=0$.

The last term $a_\eta p_\eta$, which cannot exist if $n>m$, is determined by the fact that it is given by the coördinate $\alpha_\eta = 1$, and the other coördinates, zero.

11. The difference equation 10(b) has no solution except the gamma series, since all values of $(ap)n$ are determined from it by taking $n = 1, 2, 3, \cdot\cdot$, successively. It is an equation of permanent form only for $n>m$, when it is *the general linear difference equation of* n'th *order with constant coefficients* p_1, p_2, $\cdot\cdot$, *whose general solution with m arbitarty constants is therefore found in the form of a gamma series.* The equation whose roots determine its solution (in the ordinary theory of linear difference equations) is,

$$\text{(a).}\quad x^m = p_1x^{m-1}+p_2x^{m-2}+\cdot\cdot+p_m$$

Symmetric functions Fn of the roots of this equation will also satisfy the difference equation and can therefore be expresssed as gamma series by certain values of the parameters.

Since the roots of (a) are constants, the parameters will in general be certain functions of the roots, but we propose here to determine the symmetric functions that may be expressed by gamma series *with parameters independent of the roots*; and find two sets of such functions m in each set, which can be linearly expressed in terms of each other, and either of these sets suffice to express in linear form all of the symmetric functions sought.

12. The parameter $a\eta$ of $(ap)n$, n $= 1, 2, \cdot\cdot, m$, is the coefficient of $p\eta$. Thus to determine the possible parameters of a given symmetric function, Fn, we must take a_η as the value of Fn for the roots of the equation $x^\eta = 1$, this being what 11 (a) becomes when we put $p_\eta = 1$, and other p's equal to zero. It remains to test the resulting equations,

$$F1 = a_1 p_1,\ F2 = p_1 F1 + a_2 p_2,\ F3 = p_1 F2 + p_2 F1 + a_3 p_3, \text{ etc.}$$

13. *The sum of the n'th powers*, s_η.

By art. 12, we find $a_\eta = n$, for the function s_η, and the difference equations are Newton's equations. Hence

$$s_\eta = \Sigma(1\alpha_1 \cdot\cdot n\alpha_\eta) p_1^{\alpha_1} \cdot\cdot p_\eta^{\alpha_\eta},\ \alpha_1 + \cdot\cdot + n\alpha_\eta = n$$

This is Waring's formula for s_η.

14. *The homogeneous products*, π_η.

Here, $a\eta = 1$, giving the correct difference equations,

$$\pi_1 = p_1,\ \pi_2 = p_1\pi_1 + p_2,\ \pi_3 = p_1\pi_2 + p_2\pi_2 + p_3, \text{ etc.}$$

Hence, $\pi_\eta = (1p)n$, *i. e. the coefficient of a term is the multinomial coefficient of its exponents.* Since the equations are symmetrical in π, $-p$, we have also, $p\eta = -(1[-\pi])n$. These formulas seem to be new, as also those which follow.

15. *The homogeneous products, k at a time*, πnk.

Here a_η is a binomial coefficient of the n'th power, whose value is zero for $n<k$, and 1 for $n=k$, and,

$$\pi_{nk} = (ap)n,\ a_\eta = (-1)^{k-1}(1K1.n-k.)$$

16. By applying art. 9 to the coefficients of $(ap)n$, and substituting $\pi_\eta = (1p)n$, we have

$$(a).\quad (ap)n = a_1 p_1 \pi_{\eta-1} + a_2 p_2 \pi_{\eta-2} + \cdot\cdot + a_\eta p_\eta$$

We have therefore,

	$p_1\pi_{\eta-1}$	$p_2\pi_{\eta-2}$	$p_3\pi_{\eta-3}$	$p_4\pi_{\eta-4}$	$p_5\pi_{\eta-5}$,	*etc.*
$\pi_\eta =$	1	1	1	1	1	*etc.*
$s_\eta = \pi_{\eta 1} =$	1	2	3	4	5	*etc.*
$-\pi_{\eta 2} =$		1	3	6	10	*etc.*
$\pi_{\eta 3} =$			1	4	10	*etc.*
$-\pi_{\eta 4} =$				1	5	*etc.*
$\pi_{\eta 5} =$					1	*etc.*
etc.						

From the top line and the diagonal of units, we continue adding a number to the one above for the next number in the same line (a particular case of art. 5). When $n > m$, the number of functions in each set is m.

The solution of these equations for the second set in terms of the first is found by interchanging corresponding functions, $\rho k \pi n-k$ and πnk.

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