FILTER EQUATIONS

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The equations for T-type filters are derived in several books. These derivations involve several pages of mathematical discussion. Pierce (Electrical Oscillations and Electric Waves, p. 264-323) devotes a chapter of about forty pages to the subject.

It seems to me that these equations can be derived in a much more simple manner and in a way which gives much more insight to what happens in the circuit.

The method is much the same as the solution of the equations for a transformer as is found in many text books. Starling's Electricity and Magnetism, p. 360, is one place where these equations are derived. The essential points of these transformer results are that the equivalent

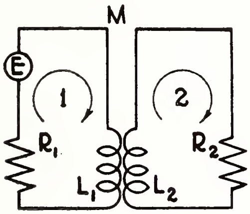

Figure 1.

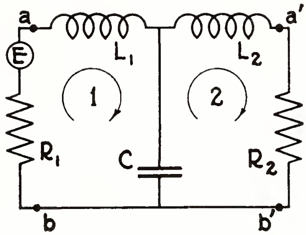

Figure 2.

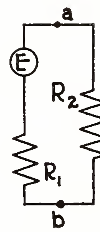

Figure 3.

Figure 1 shows the connections of an ordinary transformer in which the primary resistance is R_1 plus the equivalent resistance of R_2. Figure 3 shows a generator whose resistance R_1, connected to a load of R_2. Figure 2 shows the same generator connected to the load through a "low pass" filter. If the filter is constructed for this circuit R_2 is effectively connected direct to the generator.

primary resistance is the actual resistance of the primary plus the resistance of the secondary multiplied by the coupling factor $(M\omega/Z_2)^2$ where M is the mutual inductance and ω is $2\pi n$, n being the frequency.

The equivalent primary reactance is the primary reactance minus the secondary reactance multiplied by the same coupling factor.

A T-type low pass filter is represented by the diagram, Figure 2. In this curcuit the coupling is capacitance coupling instead of mutual inductance coupling as we have in the transformer.

The differential equations for the primary and secondary are as follows:

$$(1)\quad I_1R_2 + L_1\frac{dI_2}{dt} + \int\frac{I_1dt}{C} = Ee^{i\omega t} + \int\frac{I_2dt}{C}$$

$$(2)\quad I_2R_2 + L_2\frac{dI_2}{dt} + \int\frac{I_2dt}{C} = \int\frac{I_1dt}{C}$$

"Proc. Ind. Acad. Sci., vol. 41, 1931 (1932)."

In which we have written, for the sake of simplicity, $L_1 = L_2 = L$, L for ½L and have assumed the E.M.F. to be $Ee^{i\omega t}$, where
$E(\cos\omega t + i \sin\omega t) = Ee^{i\omega t}$.

If we let $I_1 = Ae^{i\omega t}$ and $I_2 = Be^{i\omega t}$
we get

$$(3) \quad A\left[R_1 + i\left(L\omega - \frac{1}{C\omega}\right)\right]e^{i\omega t} = \left[E - iB\frac{1}{C\omega}\right]e^{i\omega t}$$

$$(4) \quad B\left[R_2 + i\left(L\omega - \frac{1}{C\omega}\right)\right]e^{i\omega t} = -Ai\frac{1}{C\omega}e^{i\omega t}$$

Canceling out $e^{i\omega t}$ we get,

$$(5) \quad A\left[R_1 + i\left(L\omega - \frac{1}{C\omega}\right)\right] = E - iB\frac{1}{C\omega}$$

$$(6) \quad B\left[R_2 + i\left(L\omega - \frac{1}{C\omega}\right)\right] = -iA\frac{1}{C\omega}$$

Solving for the value of B in (6) we have,

$$(7) \quad B = -\frac{iA\dfrac{1}{C\omega}}{R_2 + i\left(L\omega - \dfrac{1}{C\omega}\right)}$$

Substituting this value in (5) we have,

$$(8) \quad A\left[R_1 + i\left(L\omega - \frac{1}{C\omega}\right)\right] = E - \frac{A\left(\dfrac{1}{C\omega}\right)}{R_2 + i\left(L\omega - \dfrac{1}{C\omega}\right)}$$

Rationalizing the denominator of the right hand term of (8) we have

$$(9) \quad A\left[R_1 + i\left(L\omega - \frac{1}{C\omega}\right)\right] = E - A\frac{R_2 - i\left(L\omega - \dfrac{1}{C\omega}\right)\left(\dfrac{1}{C\omega}\right)^2}{R_2^2 + \left(L\omega - \dfrac{1}{C\omega}\right)^2}$$

Collecting the reals and imaginaries we have,

$$(10) \quad A\left[R_1 + \frac{R_2\left(\dfrac{1}{C\omega}\right)^2}{R_2^2 + \left(L\omega - \dfrac{1}{C\omega}\right)^2} + i\left[L\omega - \frac{1}{C\omega}\right] - \frac{\left(\dfrac{1}{C\omega}\right)^2}{R_2^2 + \left(L\omega - \dfrac{1}{C\omega}\right)^2}\left(L\omega - \frac{1}{C\omega}\right)\right] = E$$

In this it will be seen that the equivalent resistance of the primary link of the chain is the real term and the reactance of the primary is the imaginary term.

In a balanced circuit we should have the input and the output impedances equal where the connections are as in Figure 3. Thus before the filter is inserted we have $R_1 = R_2$.

When the filter is inserted as in Figure 2, the load on the generator is R_2 if the coupling coefficient,

$$\left(\frac{1}{C\omega}\right)^2 \Big/ \left[R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2\right]$$

is made equal to unity.

Equating the coefficient to unity we have,

$$(11) \quad R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2 = \left(\frac{1}{C\omega}\right)^2$$

from which we get

$$(12) \quad R = \sqrt{\frac{2L}{C} - L^2\omega^2}$$

This is approximately equal to $\sqrt{\frac{2L}{C}}$ if the frequency is not near the cut off frequency. When the frequency becomes greater than the cut off frequency the value of R becomes imaginary. To find this critical frequency we equate the value under the radical in (12) to zero and we have,

$$(13) \quad L^2\omega^2 = \frac{2L}{C}$$

Solving for ω or $2\pi n$ we have,

$$\omega = 2\pi n = \sqrt{\frac{2}{LC}}$$

If instead of the quantity L we insert $\frac{1}{2}L$ we have,

$$(14) \quad R = \sqrt{\frac{L}{C}} \quad \text{and} \quad n = \frac{1}{2\pi} \frac{2}{\sqrt{LC}}$$

These are the usual values given for R and n.

In the imaginary term of equation (10) if the coupling coefficient is unity the reactance term becomes zero. Thus if the load in Figure 3 is a pure resistance R, the load of the generator in Figure 2 is a pure resistance, R.

From the value of R and n as given in equations (14) we get

$$L = 2R/\omega = .3183\,\frac{R}{n} \text{ Henries} \quad \text{and} \quad C = \frac{2}{\omega R} = .3183/nR \text{ Microfarads.}$$

These are the values, L and C used when one is constructing a low pass filter to be used in a circuit whose characteristic impedance is R and a cut off frequency n.

It seems to me that this derivation is much more simple, and at the same time it shows that the reactance terms of the filter cancel each other and that the resistance of the load is transferred bodily, as it were, from the second link to the first link and effectively is connected directly to the generator.