A Group of Projective Transformations Associated with a Conic Section

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The conic is variously defined in college textbooks on projective geometry as the projection of a circle on a plane, as the locus of the points of intersection of corresponding lines of two projective, non-perspective pencils in the same plane, or as the locus of the point from which four given fixed points are projected by lines making a constant cross ratio. Still another characteristic property of the conic has been used to define it, but this has been made use of rarely: first von Staudt in his *Geometrie der Lage* (1847) considered the locus of the self-conjugate points of a general plane polarity having at least one such point and proved that the locus is a conic section; in recent years Enriques used this property as his definition in his textbook on projective geometry but did not follow it consistently in his discussions. At the suggestion of Professor Tibor Radò of The Ohio State University, I undertook the problem of developing some of the projective properties of the conic from the definition in terms of a polarity and attempted to carry out systematically the plan introduced by Enriques. The present paper gives a brief summary of some of the steps in this discussion of the conic. The plan may be of interest to an advanced undergraduate class in projective geometry, perhaps as material for special or honors work in addition to the usual first course in the subject, because of the use made of some rather simple concepts not generally given consideration in elementary courses, particularly the unusual definition of the conic and the properties of projectivities in the plane and of groups of projective transformations.

The familiar properties of the conic as a curve of the second order and class follow readily from the study of self-conjugate elements in a polarity so that such considerations will be omitted from this summary. Also the general theory of poles and polars with respect to a conic is an immediate consequence of the definition of the conic itself.

As a means of securing unity in the discussion of other topics it was found convenient to associate with the conic a particular group of projective transformations. If we suppose the conic k is given as the self-conjugate points of a polarity P in the plane π, then the class of all collineations in π which transform k into itself is a group, which may be denoted as $G(P)$. Formally, the collineation C belongs to $G(P)$ if and only if

$$C^{-1} \; P \; C = P.$$

Each element of $G(P)$ either is an involution (a collineation of period two) or is a collineation which may be represented as the product of two involutions. The involutions of $G(P)$ are therefore of special interest.

Every point-and-line which are paired by the polarity in π and which are not coincident belong to a unique involution of $G(P)$, the point and line being center and axis of the involution, respectively.

Either center or axis is sufficient to determine the involution. This correspondence between points or lines of the plane and the involutions of $G(P)$ makes possible a computational method for deriving relations among the points and lines of a figure. The following theorem was found especially useful in considerations of conjugacy of points or lines:—The product of two involutions in $G(P)$ is commutative if, and only if, the centers are conjugate in P, that is, if each center lies on the axis of the other involution.

Furthermore, if two involutions are commutative, their product is a third involution, and the three involutions so related together with the identity form the four-group. The three centers are the vertices of a self-polar triangle.

The non-identical collineations of $G(P)$ belong to three classes according to the type of figure formed by their fixed elements: (a) a triangle with two vertices on the conic and a third not on the conic, the sides and vertices comprising the totality of fixed lines and points, respectively; (b) a tangent and its pole, the point of contact with the conic; (c) a line not intersecting the conic and its pole.

A general collineation of $G(P)$ has associated with it a fixed line on which are found the centers of its factor involutions and each point of which that is not a point of intersection with the conic is the center of one such factor. From the previous remark concerning commutative involutions it is seen that the factors of an involution have centers lying on the axis of the involution. This line on which the centers lie is the axis of the collineation.

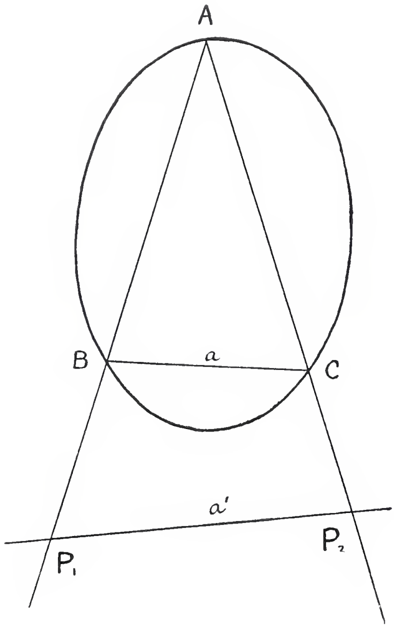
A collineation of $G(P)$ is determined uniquely when the points which correspond to an arbitrary set of three points on the conic are prescribed. There is, therefore, an isomorphism between the group $G(P)$ and the group of projectivities of points on the conic. This suggests a connection between the present treatment and the usual theory of projectivity on a conic.

A collineation is uniquely determined when an arbitrary line of the plane is prescribed as axis and the image of a single point of the conic is given, the image necessarily being chosen on the conic also, as the collineation must transform the conic into itself. Two collineations may have the same fixed elements and yet be distinct. All collineations sharing the same figure of fixed elements form a subgroup of $G(P)$.

These properties just outlined and others not listed make possible rather simple proofs of some of the standard theorems on the conic. The theorem of Pascal on the inscribed hexagon may be demonstrated by defining a collineation in $G(P)$ which carries three alternate vertices of the hexagon into the other three in a properly chosen order, and then noting that the pairs of opposite sides of the hexagon meet in points which are centers of factor involutions of the collineation and which must lie on the same line, the axis. This type of proof may be applied to the special cases of inscribed pentagon, quadrangle, and triangle without reference to the usual continuity argument.

The proof of a theorem due to von Staudt illustrates the method of proof by computations with involutions in $G(P)$.

THEOREM: If a line intersects two sides of a triangle inscribed in a conic in a pair of conjugate points, the line is conjugate to the third side, and conversely.



Let the inscribed triangle be ABC and suppose the line a' intersects AB and AC in the conjugate points P_1, P_2 respectively. Let I be the involution with axis a' and let I_1 and I_2 be those with centers P_1 and P_2 respectively. (Questions concerning the existence of these involutions, such as occur when a' is a tangent or P_1 lies on the conic, need not be considered, for in that case the proof is immediate without the use of the involutions.)

If P_1 and P_2 are conjugate, the product I_1I_2 is commutative and is an involution in $G(P)$,—in fact, the involution I since the centers lie on its axis a'. Now I carries B into C, for, in the product I_1I_2, B is taken into A by I_1 and A into C by I_2. This implies that the center of I lies on BC and hence a' is conjugate to the side $a=BC$.

The converse may be demonstrated in similar fashion.

In conclusion, attention is called to the fact that the above theorem is useful in proving the theorem of Steiner, which asserts that the points of a conic are projected from any two of its points by projective pencils of lines. This now brings the discussion into the channels followed by most texts in projective geometry.