Note on Linkages

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The article "Linkages", by Mr. Hilsenrath, in the October 1937 issue of the *Mathematics Teacher*, is one of special interest since it deals with a subject not ordinarily emphasized in mathematics courses. It opens a new and interesting study to the student with ramifications carrying over into many related fields of practical and theoretical work. The article is of greatest interest if the reader actually constructs the linkages described and observes their operation. The construction of these linkages, however, presents some problems because of a few slight inaccuracies and omissions in the article.

The most serious omission appears to occur in the discussion of the Peaucellier Conicograph. It is stated in the article that when the length of the

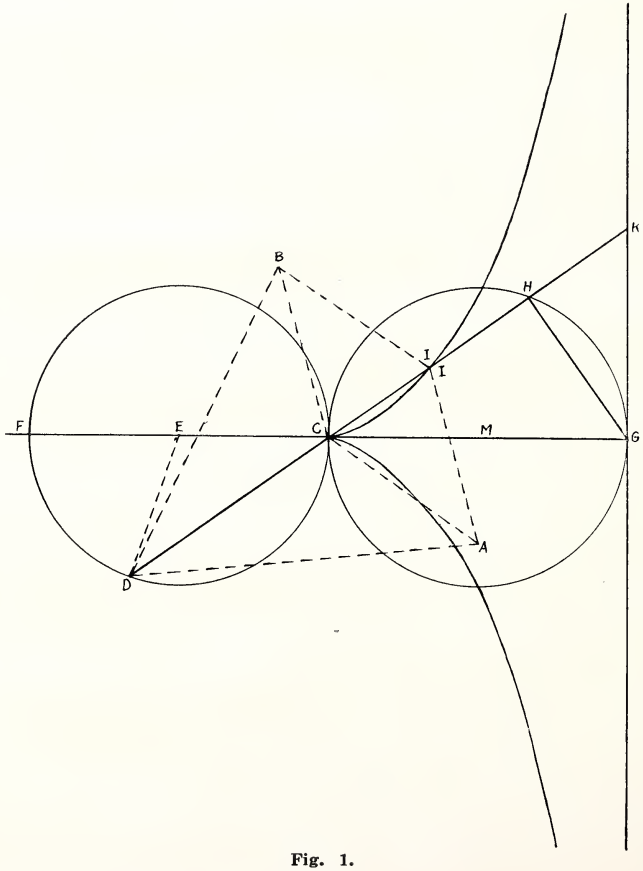

Fig. 1.

radius bar ED is equal to the distance between E and C, then the curve described by I is a Cissoid of Diocles. This is a necessary but not a sufficient condition. Evidently there are an infinite number of positions of E that will enable us to have a radius bar $ED=EC$ (Fig. 1), but in only one of these positions will the cissoid be drawn. The condition that Mr. Hilsenrath has omitted is this:

$$EC = \tfrac{1}{2}\sqrt{\overline{DA}^2 - \overline{AI}^2}$$

As pointed out by Mr. Hilsenrath, the most important property of the Peaucellier Cell (used in this linkage with C as the fixed point instead of D) is that the product $DC \cdot DI = \overline{DA}^2 - \overline{AI}^2 =$ a constant. When the rhombus is collapsed so that C and I coincide, then we readily see that

$$(1) \qquad DC \cdot DI = \overline{DC}^2 = \overline{FC}^2.$$

FC then equals $2EC$, and is the diameter of the circle which the point D must follow if the linkage is to draw the cissoid.

Now let another circle with center M be drawn tangent to the first circle at C and with diameter $CG=FC$. Then

$$CH = DC.$$

Also $$DI = DC + CI.$$

Substituting these values in equation (1) we get

$$(2) \qquad CH\,(CH + CI) = \overline{CG}^2.$$

The question now is, does this last equation satisfy the requirements of the cissoid? Let GK be tangent to the circle at G. Lay off CI' equal to HK. Then by definition the locus of the point I' is a cissoid. By using the triangles CHG and CKG, we note the following facts:

$$\angle HCG = \angle KCG$$
$$\angle CHG = \angle CGK$$
$$\therefore \triangle CHG \sim \triangle CKG$$
$$\therefore \frac{CH}{CG} = \frac{CG}{CK}$$

or $$CH \cdot CK = \overline{CG}^2.$$

But $$CK = CH + HK = CH + CI'$$

$$(3) \qquad \therefore CH\,(CH + CI') = \overline{CG}^2.$$

From a comparison of equations (2) and (3) it is evident that I and I' must be the same point, and, therefore, when the linkage is deformed, the point I describes the cissoid.

The discussion of Bricard's Straight-Line Motion also appears to contain some slight inaccuracies. The article states that the length $DB=EC=\frac{a^2}{b}$, and that $DE=\frac{ac}{b}$. But these two conditions are evidently impossible from the appearance of the linkage and they do not seem to fit the proof needed to establish the theory of the instrument. The correct ratios appear to be these: $DB=EC=\frac{b^2}{a}$, $DE=\frac{bc}{a}$.

Mr. Hilsenrath also states that the proof for this linkage is based on the theorem that a straight line is the locus of a point which moves so that the difference between the squares of its distances from two fixed points is a constant. The theorem best applied here, however, seems to be a much simpler

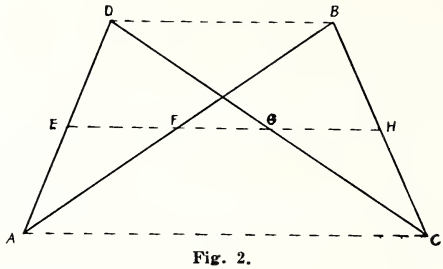

Fig. 2.

one from elementary geometry, namely: The locus of a point equidistant from two fixed points is a straight line which is the perpendicular bisector of the line joining the fixed points. Using this theorem and the ratios given above, the proof for this linkage is quite simple. By drawing the lines FA, DA, etc., we form triangles. It may then be proved in several ways, using congruency or similarity of triangles that $FA = AG$. According to the above theorem, then, the locus of A is a straight line.

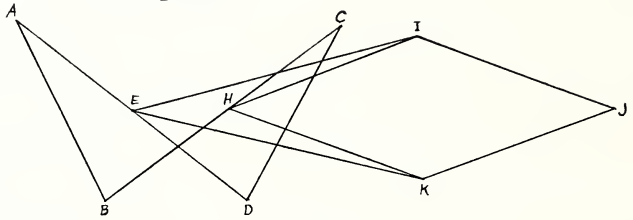

Fig. 3

The Peaucellier Cell is of interest, of course, primarily because it was the first known means of drawing a straight line without the use of a straight edge. The Cell, however, demonstrates so effectively the basic facts of the geometry of inversion that one seems to be wasting a valuable opportunity if he does not construct this linkage so as to make it do more than merely illustrate the straight line. After we have constructed our Cell, then the radius of the circle of inversion of the Cell is $r = \sqrt{\overline{DA}^2 - \overline{AI}^2}$. This circle may be drawn on the board, and it helps immeasurably in understanding the definition of inversion. Now if we let C or I follow a straight line through D, it becomes evident that any line through the center of inversion is its own inverse. As indicated by Mr. Hilsenrath, a circle through D inverts into a straight line, and a circle not through D inverts into a circle. It is possible, however, to fix the center and radius of this second circle so that both C and I describe complete circles instead of arcs, and this illustrates the theorem vividly. Also it is worth while to arrange that C can draw still a third circle orthogonal to the circle of inversion. Then we note that I follows the same circle; that is, a circle orthogonal to the circle of inversion is its own inverse.

As Mr. Hilsenrath effectively points out, the nine linkages described in his article do not exhaust the subject of linkages. There are a number of other linkages which are sufficiently simple that they might well be added to a collection. Two of these are here described.

It is interesting to note that the Peaucellier Cell is not the only instrument which has the property that the product of the distances of two points from a

third point is a constant. A simpler arrangement is the inversor by Hart illustrated in Figure 2. $ABCD$ is a contra-parallelogram and E, F, G, and H are the mid-points of the sides. Now it is evident from the figure that:

$$EG = FH = \tfrac{1}{2}AC$$
$$EF = GH = \tfrac{1}{2}DB$$

It is also evident that A, D, B, and C are concyclic. Therefore, by Ptolemy's theorem,

$$AC \cdot DB + AD \cdot BC = AB \cdot DC$$

But $AD = BC$ and $AB = DC$

$$\therefore AC \cdot DB = \overline{AB}^2 - \overline{AD}^2$$

$$\tfrac{1}{2}AC \cdot \tfrac{1}{2}DB = \tfrac{1}{4}(\overline{AB}^2 - \overline{AD}^2)$$

$$\therefore EG \cdot EF = \tfrac{1}{4}(\overline{AB}^2 - \overline{AD}^2) = \text{a constant}$$

Therefore, if we fix the point E, and F describes a circle through E, then G draws a straight line. That is, exactly the same things that are accomplished by the Peaucellier Cell are also accomplished by this instrument. It is not actually necessary that E, F, G, and H be the mid-points of the sides of the contra-parallelogram. It is only necessary that they lie on a straight line parallel to AC.

An interesting variation of the above linkage is had by fixing the side AD and letting H be the marking point. A variety of curves can then be drawn, depending on the ratio between the sides AD and AB. If we now attach a Peaucellier Cell to the linkage so that the point connecting the long sides of the Cell is fixed at E and the inner point of the rhombus is attached to H, then the outer point of the rhombus evidently draws the inverse of the curve described by H. (As shown above, a second contra-parallelogram could be used instead of the Cell with the same results.) A very interesting special case of this linkage is represented in Figure 3. Let the contra-parallelogram be constructed so that $AD : AB :: \sqrt{2} : 1$. AD thus becomes the long side, and E and H are the mid-points of AD and BC respectively. Let AD be the fixed side. It can be proved by analytical geometry that when this linkage is deformed the locus of H is a lemniscate. Now if a Peaucellier Cell is attached to the points E and H, then J describes the inverse of the lemniscate. That is, J describes an equilateral hyperbola.