The Solution of Bernoulli's Differential Equation by Means of Integrating Factors

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The solution of differential equations by means of integrating factors is seldom treated very satisfactorily in textbooks for beginners. The probable reason is that the theoretical development of integrating factors involves a linear partial differential equation. However, students beginning differential equations are, or should be, familiar with elementary partial differentiation, and this assumption will be made in the developments given here. The purpose of this paper is to point out the unity of method and insight into the nature of integrating factors that may be obtained by elementary methods and to secure the solution of Bernoulli's differential equation by means of an integrating factor as a fine illustration of the method.

The usual form of the first order differential equation is

$$Mdx+Ndy=0, \tag{1}$$

where M and N are functions of either x or y or both. In general the equation is not exact, and, although theoretically the number of integrating factors is infinite, the determination of even one such factor is frequently beyond the powers of the beginner, even if considered practicable. However, suppose $Z(x,y)$ to be an integrating factor of $Mdx+Ndy=0$. Then

$$ZMdx+ZNdy=0 \tag{2}$$

is exact, and the condition for exactness

$$\frac{\partial(ZM)}{\partial y}=\frac{\partial(ZN)}{\partial x} \tag{3}$$

is satisfied and one readily obtains

$$Z\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = N\frac{\partial Z}{\partial x}-M\frac{\partial Z}{\partial y}$$

which is of the form $Pp+Qq=R$, the first order partial differential equation.

By assuming Z to be a function of x only, or of y only, or of the form $e^{f(x)}$ or $e^{f(y)}$, one obtains the usual special forms of integrating factors. Thus, for example, consider the linear differential equation $dy+(Py-Q)dx=0$, where P and Q are functions of x only. Multiplying by Z, one has $Zdy+Z(Py-Q)dx=0$. which is now exact, and, because the equation is exact, equation (3) becomes

$$\frac{\partial Z}{\partial x}=\frac{\partial Z}{\partial y}(Py-Q)+PZ.$$

If now Z is assumed to be a function of x only, one has $dZ/Z=Pdx$, whence $Z=e^{\int Pdx}$, the well known form of the integrating factor. As another illustration consider the equation $y(2xy+e^x)dx-e^xdy=0$. Since y appears to a higher power than the first and since e^x appears, assume $Z=e^{f(x)}y^m$. Then

$$e^{f(x)}y^m(2xy+e^x)dx-e^{f(x)+x}y^mdy=0$$

is exact, and applying equation (3) and simplifying, one has

$$(m+2)2xy+e^x[m+2+f'(x)]=0$$

identically, whence $m+2=0$, and $m=-2$, and $f(x)$ is constant. Accordingly $Z=y^{-2}$ is the integrating factor. As another illustration consider the equation

$$x^2y\,dy+(y^2-x^4)dx=0.$$

Let $Z=e^{f(x)}y^n$. Then

$$e^{f(x)}y^{n+1}x^2dy+e^{f(x)}(y^{n+2}-x^4y^n)dx=0$$

is exact and equation (3) becomes, after simplification,

$$y^2[2x+x^2f'(x)-(n+2)]+nx^4=0.$$

Therefore $n=0$, and $2x+x^2f'(x)-2=0$, whence $f'(x)=2/x^2-2/x$, $f(x)=-2/x-2\log x$, and $Z=e^{-\frac{2}{x}}/x^2$.

As a final illustration consider

$$xyt^2(3y\,dx+x\,dy)-(2y\,dx-x\,dy)=0.$$

Let $Z=x^my^n$, whence, upon multiplying through by Z, applying equation (3), and simplifying, one has

$$(3n-m+7)xy^2-(2n+m+3)=0,$$

and $3n-m+7=0$, $2n+m+3=0$. Solving, one finds $m=1$, $n=-2$, and $Z=x/y^2$.

Bernoulli's equation is

$$dy+(Py-Qy^n)dx=0,$$

where P and Q are functions of x only. Assume $Z=e^{f(x)}y^k$. Multiplying the equation by Z, applying equation (3) and simplifying, one has

$$f'(x)=k(P-Qy^{n-1})+P-nQy^{n-1}.$$

Separating the variables,

$$[(k+1)P-f'(x)]/_Q=(k+n)y^{n-1}.$$

The left member of this last equation is a function of x only, and the right member is a function of y only. Hence $k+n=0$, whence $k=-n$, and then

$$(1-n)P-f'(x)=0,$$

whence $f(x)=\int(1-n)Pdx$, and $Z=y^{-n}e^{\int(1-n)Pdx}$. Thus one may write down the integrating factor immediately whereas the usual method of solution is to make a substitution that will reduce the equation to the linear form.