Symmetric Jacobi Polynomials

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1. Introduction. It is felt advisable to make more readily available the formulas one finds necessary in working with series of symmetric Jacobi polynomials. These polynomials are denoted by $F(n+p,-n,(p+1)/2,(1-x)/2)$, $G_n(p,(p+1)/2,(1-x)/2)$, $X_n^{\frac{p-1}{2},\frac{p-1}{2}}(x)$ and, more recently, $X_n^{\frac{p-1}{2}}(x)$. In mathematical literature most developments are made with two members of the family, Tschebychef (trigonometric) polynomials, $p=0$, and Legendre polynomials, $p=1$. Legendre polynomials are easiest to use. Developments can and should be made for the family rather than particular members in certain cases.

2. History. Gauss[1] was interested in the Hypergeometric Series

$$F(\alpha,\beta,\gamma,x)=1+\frac{\alpha\cdot\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)}{1\cdot 2}\,\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2+\cdots\cdot$$

$$\tag{1} \cdots+\frac{\alpha(\alpha+1)\cdots\cdot(\alpha+k-1)}{1\cdot 2\cdots\cdots k}\cdot\frac{\beta(\beta+1)\cdots\cdots(\beta+k-1)}{\gamma(\gamma+1),\cdots\cdots(\gamma+k-1)}x^k+\cdots\cdot$$

as the solution of the differential equation

$$\tag{2} x(1-x)y''+[\gamma-(\alpha+\beta+1)x]y'-\alpha\beta y=0.$$

Jacobi[2] discovered that this series terminates itself if one of the elements α or β, which enter the series symmetrically, is a negative integer. The polynomials are solutions of the differential equation, but can also be obtained by successive differentiation of a function of x.

$$\tag{3} \begin{aligned} &F(\alpha+n,-n,\gamma,x)\\ &=1+\frac{(\alpha+n)(-n)}{1\cdot\gamma}x+\frac{(\alpha+n)(\alpha+n+1)}{1\cdot 2}\cdot\frac{(-n)(-n+1)}{\gamma(\gamma+1)}x^2+\cdots\cdots\\ &\cdot\cdot+\frac{(\alpha+n)(\alpha+n+1)\cdots(\alpha+2n-1)}{1\cdot 2\cdot 3\cdots\cdot n}\cdot\frac{(-n)(-n+1)\cdots(-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)}x^n.\end{aligned}$$

$$\tag{4} \begin{aligned} &F(\alpha+n,-n,\gamma,x)\\ &=\frac{x^{1-\gamma}(1-x)^{\gamma-\alpha}}{\gamma(\gamma+1)\cdots\cdot(\gamma+n-1)}\,\frac{d^n}{dx^n}\,x^{n+\gamma-1}(1-x)^{\alpha+n-\gamma}.\end{aligned}$$

Darboux[3] and Abramescu[4] wrote extensive articles on Jacobi polynomials. Darboux's article was the source of most of the formulas in this paper. He it

[1]Gauss, 1812. Disquisitiones generales Circa Seriem Infinitam, Werke, 3:127.

[2]Crelle's Journal, 1859. 56:149-165.

[3]Mémoire sur l'approximation des fonctions de très grands nombres. Journal de Math. 1878. 4:5-60, 377-416.

[4]Sulle serie di polinomi di una variable complessa. Le serie di Darboux. Annali di Mathematica. 31:207-249.

was who first extended the field of orthogonal polynomials from the real axis to the complex plane.

3. Transformation for Symmetric Polynomials. The condition $\alpha-\gamma=\gamma-1$, where the difference between α and γ is equal to the difference between γ and *1*, and the transformation of x to $(1-x)/2$ gives "symmetric" Jacobi polynomials. They cause

1. Alternate coefficients to become zero, thereby eliminating much work in calculations.
2. The recursion formula to include only terms of first degree in n. This was necessary in the proof of Cesàro summability of $\sum_1^{\infty} a_n n^p X_n$, where p is a positive integer[5].
3. The range to be $-1<x<1$, instead of $0<x<1$.

For Tschebychef polynomials $\alpha=0$, $\gamma=\frac{1}{2}$, the difference being $\frac{1}{2}$. For Legendre polynomials $\alpha=\gamma=1$, the difference being *0*.

There is no reason why α cannot be greater than *1*, ie. $\alpha=3$, $\gamma=2$, or $\alpha=5$, $\gamma=3$. The only rigid condition the writer finds is $\alpha>-1$.

4. The Differential Equation. The original hypergeometric differential equation of Gauss (2) is transformed to

$$(1-x^2)X_n''-[(p+1)x]X_n'+(p+n)nX_n=0. \tag{5}$$

In modern notation $\alpha=p$ and $\gamma=(p+1)/2$, one choosing any $p>-1$. Unless explicitly stated otherwise the formulas throughout the remainder of the paper, including (5), all contain p, X_n denotes $X_n^{\frac{p-1}{2}}(x)$, and x is confined to the range $-1<x<1$.

5. The Recursion Formula. From the recursion formula

$$xX_n=\frac{n+p}{2n+p}X_{n+1}+\frac{n}{2n+p}X_{n-1} \tag{6}$$

one can get any set of polynomials by taking $X_0=1$, $X_1=x$, and assuming a value of $p>-1$.

The polynomials can be obtained from (3) or (4), letting $x=(1-x)/2$, $\alpha=p$ and $\gamma=(p+1)/2$. They are also obtained from the generating function.

6. The Generating Function. Brenke's method[6] of deriving the generating function is based on the Lagrange expansion formula[7].
Putting the differential equation (5) in the form

$$\frac{1-x^2}{p+1}X_n''-xX_n'+\frac{p+n}{p+1}nX_n=0,$$

one substitutes the coefficient of X_n'' for $\phi(y)$, changing x to y, in the equation $y=x+t\phi(y)$ and obtains

$$y=x+t\frac{1-y^2}{p+1}. \tag{a}$$

[5]Cowgill, 1935. On the summability of a certain class of series of Jacobi Polynomials. Bull. Am. Math. Soc. 41: 541-549.

[6]On polynomial solutions of a class of linear differential equations of the second order. Bull. Am. Math. Soc. 36:77-84, 82. 1930.

[7]Goursat-Hedrick, 1904. Mathematical Analysis. Vol. 1, par. 189.

Differentiating partially with respect to x

$$\frac{\partial y}{\partial x}=1+\frac{t}{p+1}(-2y)\frac{\partial y}{\partial x}\ ,\quad \frac{\partial \gamma}{\partial \chi}=\frac{1}{1+\frac{2ty}{p+1}}$$

From (a)

$$\frac{ty^2}{p+1}+y-\left(x+\frac{t}{p+1}\right)=0.$$

Solving by the quadratic formula and making the transformation $t=-\frac{p+1}{2}t$,

$$y=\frac{-1\pm\sqrt{1-2tx+t^2}}{-t}\ .$$

$$\frac{\partial y}{\partial x}=\frac{1}{1-ty}=\frac{1}{\sqrt{1-2tx+t^2}}.$$

Substituting the "characteristic function"

$$\rho=(p+1)(1-x^2)^{\frac{p-1}{2}}$$

in Brenke's expression for the generating function,

(7) $$\psi(x,t)=\frac{\rho(y)}{\rho(x)}\frac{\partial y}{\partial x}=\frac{(p+1)(1-y^2)^{\frac{p-1}{2}}}{(p+1)(1-x^2)^{\frac{p-1}{2}}}\frac{1}{\sqrt{1-2tx+t^2}}$$

$$=\left[\frac{t^2-(1-2\sqrt{1-2tx+t^2}+1-2tx+t^2)}{t^2(1-x^2)}\right]^{\frac{p-1}{2}}\frac{1}{\sqrt{1-2tx+t^2}}$$

$$=\frac{\left[\frac{4}{1-x^2}(2tx-2+2\sqrt{1-2tx+t^2})\right]^{\frac{p-1}{2}}}{(2t)^{p-1}\sqrt{1-2tx+t^2}}$$

$$=\sum_{r=0}^{\infty}a_rX_rt^r\ ,$$

where

$$a_n=\frac{(p+1)(p+3)\ \cdots\cdots\ (p+2n-1)}{2^n\ n!}$$

$$=\frac{\Gamma((p+1)/2+n)}{\Gamma((p+1)/2)\Gamma(n+1)}=0\left(n^{\frac{p-1}{2}}\right).$$

$$X_n=0\left(n^{-\frac{p}{2}}\right),\ \text{so}\ a_nX_n=0\left(n^{-\frac{1}{2}}\right)$$

$$X_n(1)=1,\ X_n(-1)=(-1)^n\ [8]$$

[8]This is shown by using equations (1) and (4), Darboux, loc. cit., p. 377, and making the transormation $x=(1-\xi)/2$, $x=0$ corresponding to $\xi=1$ and $x=1$ corresponding to $\xi=-1$, $\alpha-\gamma=\gamma-1$.

One also has the relation[9]

$$(8)\qquad \sum_{r=0}^{\infty} \frac{\Gamma(r+p)}{\Gamma(p)\,\Gamma(r+1)} X_r t^r = \frac{1}{(1-2tx+t^2)^{\frac{p}{2}}} \qquad (0<t<1).$$

Differentiating both sides of (8), multiplying throughout by $2t$ and adding to (8), one gets

$$\sum_{r=0}^{\infty} \left[\frac{\Gamma(r+p)}{\Gamma(p)\,\Gamma(r+1)}\right] (2r+p)\, X_r\, t^r = \frac{p(1-t^2)}{(1-2tx+t^2)^{\frac{p}{2}+1}}$$

which one can write in the form

$$(9)\qquad \sum_{r=0}^{\infty} \left[\frac{\Gamma(r+p)}{\Gamma(r+1)}\right] (2r+p)\, X_r\, t^r = \frac{\Gamma(p+1)}{(1-2tx+t^2)^{\frac{p}{2}}} \, \frac{1-t^2}{1-2tx+t^2}.$$

Let $p=0$ and one obtains the special generating function for the Tschebychef polynomials

$$(10)\qquad \sum_{r=0}^{\infty} 2[X_r^{-\frac{1}{2}}(x)]\, t^r = \frac{1-t^2}{1-2tx+t^2}, \text{ or}$$

$$\sum_{r=0}^{\infty} T_r(x)(2t)^r = \frac{1-t^2}{1-2tx+t^2} \text{ where } T_n(x) = \frac{1}{2^{n-1}}[X_n^{-\frac{1}{2}}(x)].$$

One can now see that series (9) is the Cauchy product of series (8) and series (10), as well as being the right hand member of the Christoffel-Darboux identity

$$(11)\qquad \frac{\Gamma(n+p)}{\Gamma(p)\Gamma(n+1)}(n+p)\frac{X_{n+1}-X_n}{x-1}$$

$$= \sum_{r=0}^{n} \frac{\Gamma(r+p)}{\Gamma(p)\,\Gamma(r+1)}(2r+p)X_r, \text{ when } t=1.$$

7. Symmetric Jacobi Polynomials.

$p=0$, Tschebychef Polynomials.

$X_0(x)=1$	$X_0(\cos\theta)=1$
$X_1(x)=x$	$X_1(\cos\theta)=\cos\theta$
$X_2(x)=2x^2-1$	$X_2(\cos\theta)=\cos 2\theta$
$X_3(x)=4x^3-3x$	$X_3(\cos\theta)=\cos 3\theta$
$X_4(x)=8x^4-8x^2+1$	$X_4(\cos\theta)=\cos 4\theta$

$p=\frac{1}{2}$.

$X_0=1$	$X_3=3(x^3-\frac{2}{3}x)$
$X_1=x$	$X_4=\frac{1}{7}(39x^4-36x^2+4)$
$X_2=\frac{5}{3}(x^2-\frac{2}{5})$	

[9]Cowgill, *loc. cit.*, pp. 543 and 545.

p=1. Legendre Polynomials.

$X_0=1$ $\qquad$ $X_3=\frac{1}{2}(5x^3-3x)$

$X_1=x$ $\qquad$ $X_4=\frac{1}{8}(35x^4-30x^2+3)$

$X_2=\frac{1}{2}(3x^2-1)$

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p=3.

$X_0=1$ $\qquad$ $X_3=\frac{1}{4}(7x^3-3x)$

$X_1=x$ $\qquad$ $X_4=\frac{1}{8}(21x^4-14x^2+1)$

$X_2=\frac{1}{4}(5x^2-1)$

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p=5.

$X_0=1$ $\qquad$ $X_3=\frac{1}{2}(3x^3-x)$

$X_1=x$ $\qquad$ $X_4=\frac{1}{16}(33x^4-18x^2+1)$

$X_2=\frac{1}{6}(7x^2-1)$

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8. **The Orthogonality Property.** In expanding any function of x into an infinite series of symmetric Jacobi polynomials it is necessary that the polynomials have the orthogonality property expressed by

$$\int_{-1}^{1}\rho X_m X_n dx=\begin{cases}0\ , & m \text{ not equal to } n\\ J_n, & m=n\ ,\end{cases} \tag{12}$$

so that the coefficients can be determined.

Brenke's[10]; condition for orthogonality will hold with the polynomial solutions of (5).

The derivation of the characteristic function ρ and Darboux's value of the constant J_n are best explained in detail.

9. **The Characteristic Function** ρ. To explain the derivation of the characteristic function ρ one assumes the differential equation

(b) $\qquad rX_n''+sX_n'+t_nX_n=0$, t_n a function of n.

Multiplying (b) by ρ so as to make it "self adjoint", one has

(c) $\qquad \rho rX_n''+\rho sX_n'+\rho t_nX_n=0$, where

(d) $\qquad \frac{d}{dx}\rho r=\rho s=\rho' r+\rho r'$. (b) can now be written

(e) $\qquad \frac{d}{dx}(\rho rX_n')+\rho t_nX_n=0.$

From (d) $\rho' r=\rho(s-r')$. $\frac{\rho'}{\rho}=\frac{s-r'}{r}=\frac{s}{r}-\frac{r'}{r}.$

Multiplying by dx and integrating

$$\int\frac{\rho'}{\rho}dx=\int\frac{s}{r}dx-\int\frac{r'}{r}dx\ ,$$

[10]Brenke, *loc. cit.*, Case 1, pp. 78-79.

$$\log \rho = \int \frac{s}{r} dx - \log r \, ,$$

$$\log \rho + \log r = \log \rho r = \int \frac{s}{r} dx \, ,$$

$$\rho r = e^{\int \frac{s}{r} dx} \quad \text{or} \quad \rho = \frac{1}{r} e^{\int \frac{s}{r} dx}$$

Taking the differential equation (5) in the form

$$\frac{1-x^2}{p+1} X_n'' - xX_n' + \frac{(p+n)n}{(p+1)} X_n = 0$$

and comparing it to (b)

$$r = \frac{1-x^2}{p+1} \text{ and } s = -x, \quad \text{so}$$

(13) $$\rho = \frac{p+1}{1-x^2} e^{(p+1)\int \frac{-xdx}{1-x^2}} = (p+1)(1-x^2)^{\frac{p-1}{2}} .$$

Write the "self adjoint" equations for X_n and X_m.

(f) $$\frac{d}{dx}(\rho r X_n') + \rho t_n X_n = 0$$

(g) $$\frac{d}{dx}(\rho r X_m') + \rho t_m X_m = 0 \, .$$

Multiply (f) by X_m and (g) by X_n. Subtracting, multiplying by dx and integrating[11] one gets

$$(t_m - t_n)\int_{-1}^{1} \rho X_m X_n dx = \int_{-1}^{1} X_m \frac{d}{dx}\left(\rho r \frac{dX_n}{dx}\right) dx - \int_{-1}^{1} X_n \frac{d}{dx}\left(\rho r \frac{dX_m}{dx}\right) dx \, .$$

Integrating by parts

$$(t_m - t_n)\int_{-1}^{1} \rho X_m X_n dx = \left[X_m\left(\rho r \frac{dX_n}{dx}\right) - X_n\left(\rho r \frac{dX_m}{dx}\right)\right]_{-1}^{1}$$

$$- \int_{-1}^{1} \rho r \frac{dX_n}{dx}\frac{dX_m}{dx} dx + \int_{-1}^{1} \rho r \frac{dX_m}{dx}\frac{dX_n}{dx} dx \, .$$

As ρr contains the factor $(1-x^2)$ the non-integral term is 0 and $\int_{-1}^{1} \rho X_m X_n dx = 0$ unless $m = n$. This last equation expresses the orthogonality property (12).

10. Derivation of J_n. The writer found it necessary to solve for J_n with the notation of Darboux and general Jacobi polynomials. From Darboux[12] "One has also

$$J_n = \int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} X_n^2 dx$$

$$= \frac{\Gamma(n+1) \; \Gamma^2(\gamma) \; \Gamma(\alpha+n-\gamma+1)}{(2n+\alpha) \; \Gamma(\alpha+n) \; \Gamma(\gamma+n)} \text{''} .$$

[11] Byerly, Fourier's Series. Par. 91, p. 171.

[12] Loc.cit., (44), p. 46 and (13), p. 22,

and

$$\text{“}X_n = F(\alpha+n, -n, \gamma, x) = \frac{x^{1-\gamma}(1-x)^{\gamma-\alpha}}{\gamma(\gamma+1)\cdots\cdots(\gamma+n-1)} \frac{d^n}{dx^n} x^{n+\gamma-1}(1-x)^{\alpha+n-\gamma}\text{”}.$$

$$\Gamma(\gamma+n) = (\gamma+n-1)(\gamma+n-2)\cdots\cdots(\gamma+1)\gamma\Gamma(\gamma),$$

so

$$\gamma(\gamma+1)\cdots\cdots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}.$$

$$J_n = \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)} \int_0^1 [X_n] \left[\frac{d^n}{dx^n} x^{n+\gamma-1}(1-x)^{\alpha+n-\gamma}\right] dx.$$

One applies integration by parts n times, where $\int u\,dv = uv - \int v\,du$, letting the polynomial always be u and the derivative and dx be dv. The uv term is always zero with the limits 0 and 1, as it always contains the terms x and $(1-x)$ to some power. Differentiating X_n, (3), n times one finishes with the coefficient of x^n multiplied by $n!$, so

$$\frac{d^n}{dx^n} X_n = \Gamma(n+1)\frac{\Gamma(\alpha+2n)}{\Gamma(n+1)\Gamma(\alpha+n)} \frac{(-1)^n\Gamma(n+1)\Gamma(\gamma)}{\Gamma(\gamma+n)}$$

$$= (-1)^n \frac{\Gamma(\alpha+2n)\Gamma(n+1)\Gamma(\gamma)}{\Gamma(\alpha+n)\Gamma(\gamma+n)}.$$

After n integrations by parts

$$J_n = \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)}(-1)^{2n}\frac{\Gamma(\alpha+2n)\Gamma(n+1)\Gamma(\gamma)}{\Gamma(\alpha+n)\Gamma(\gamma+n)} \int_0^1 x^{n+\gamma-1}(1-x)^{\alpha+n-\gamma}dx.$$

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$ [13]

so

$$\int_0^1 x^{n+\gamma-1}(1-x)^{\alpha+n-\gamma}dx = \frac{\Gamma(\gamma+n)\Gamma(\alpha+n-\gamma+1)}{\Gamma(2n+\alpha+1)}.$$

$$J_n = \frac{\Gamma(n+1)\Gamma^2(\gamma)\Gamma(\alpha+n-\gamma+1)}{(2n+\alpha)\Gamma(\alpha+n)\Gamma(\gamma+n)},$$ Darboux's result.

Making the transformation to symmetric Jacobi polynomials

(14) $$Jn = \int_{-1}^{1}(p+1)(1-x^2)^{\frac{p-1}{2}}\left[X_n^{\frac{p-1}{2}}(x)\right]^2 dx$$

$$= \frac{2^p(p+1)\Gamma(n+1)\Gamma(2^{\frac{p-1}{2}})}{(2n+p)\Gamma(p+n)}$$ [14].

11. The Derivative Form. Transforming (4) one obtains

(15) $$X_n = \frac{(-1)^n(1-x^2)^{\frac{1-p}{2}}}{(p+1)(p+3)\cdots\cdots(p+2n-1)} \frac{d^n}{dx^n}(1-x^2)^{\frac{p+2n-1}{2}}.$$

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[13] Woods, Advanced Calculus, p. 166.

[14] See Byerly, loc.cit., par. 89, p. 168, for derivation of Jn for Legendre polynomials.