A Simple Method for Obtaining the Solutions of Dirac's Equation for a Free Particle in Spherical Coordinates

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The Dirac wave equation for a free electron.

$$(E-c(\vec{\alpha}\cdot\vec{P}-\beta mc^2)\Psi=0 \tag{1}$$

may for convenience be written in the form

$$(\epsilon+i\vec{\alpha}\cdot\nabla-\beta)\Psi=0 \tag{2}$$

where the units are chosen so that $\frac{h}{2\pi}=\frac{1}{2\pi}$ times Planck's constant; m, the mass of the electron, and c, the velocity of light, are put equal to unity. In these units the unit of length is $\frac{h}{2\pi mc}$, the unit of time is h/mc^2, and the unit of mass is the electron mass. ϵ is the total energy in these units, $-i\nabla$ the operator representing the momentum, and $\vec{\alpha}$ and β the four row and column Dirac spin matrices.

Let us call the operator $\epsilon+i\vec{\alpha}\cdot\nabla-\beta=H$ and define the operator $K=\epsilon-i\vec{\alpha}\cdot\nabla+\beta$. Consider the product HK. By using the well-known properties of the α's and β, namely,

$$\begin{aligned}&\alpha_X\alpha_Y+\alpha_Y\alpha_X=0,\text{ etc.}\\&\alpha_X\beta+B_X=0,\text{ etc.}\\&\alpha_X=\alpha_Y^2=\alpha_{2Z}=\beta^2=1\end{aligned} \tag{3}$$

we find for the product of H and K that

$$HK=\nabla^2+\epsilon^2-1 \tag{4}$$

where ∇^2 is the Laplacian. Defining $\epsilon^2-1=k^2$, we have $HK=\nabla^2+k^2$. Now consider the partial differential equation

$$(\nabla^2+k^2)\Phi=0 \tag{5}$$

and let us express ∇^2 in spherical polar coordinates. The solutions Φ of this equation are well-known.

$$\Phi=\frac{R(r)}{r^{1/2}}S(\vartheta,\phi) \tag{6}$$

where R(r) is a Bessel function and $S(\vartheta,\phi)$ is a spherical harmonic. Now let us return to our original equation, $H\Psi=0$ (2).

If we put $\Psi=K\Phi$ we obtain the differential equations $HK\Phi=0$ (7)
so that $\Psi=K\Phi$ is a solution of the Dirac equation for a free particle in spherical polar coordinates. Since the operator K is a matrix, each column of the matrix gives us the four components of the solution.

The four columns, then, are the four independent solutions corresponding to the two independent orientations of the spin and to the two possible values of the energy associated with a given value of the momentum.