THE PRODUCT OF TWO INTEGRALS

WILL E. EDINGTON, Purdue University

Consider two integrals $\int P(x)dx$ and $\int Q(x)dx$, where $P(x)$ and $Q(x)$ are continuous functions of x possessing derivatives except possibly at a finite number of points. The following is a preliminary study of the product of these two integrals when the following relation is satisfied:

$$\int P(x)dx \int Q(x)dx = k\int P(x)Q(x)dx. \tag{1}$$

k is an arbitrary constant which may be given suitable values in certain developments following that will simplify the results. For the sake of brevity P and Q will be used to represent the functions P(x) and Q(x) respectively, and P' and Q' will stand for their first derivatives.

Differentiating both members of (I) with respect to x gives

$$P\int Qdx + Q\int Pdx = kPQ. \tag{2}$$

Differentiating both members of (2) with respect to x gives finally

$$P'\int Qdx + Q'\int Pdx = k(P'Q+PQ') - 2PQ. \tag{3}$$

Solving for the integrals in (2) and (3) gives

$$\int Pdx = \frac{P^2(2Q-kQ')}{P'Q-PQ'}, \qquad \int Qdx = \frac{Q^2(kP'-2P)}{P'Q-PQ'}. \tag{4}$$

Clearing the first expression of (4) of fractions and rearranging the terms gives

$$P'(kQ^2 - Q\int Qdx) = P(2Q^2 - Q'\int Qdx).$$

whence $$\frac{P'}{P} = \frac{2Q^2 - Q'\int Qdx}{kQ^2 - Q\int Qdx} \text{ and } \log P = \int \frac{2Q^2 - Q'\int Qdx}{kQ^2 - Q\int Qdx}. \tag{5}$$

Likewise $$\log Q = \int \frac{2P^2 - P'\int Pdx}{kP^2 - P\int Pdx}. \tag{6}$$

The expressions (5) and (6) are in such a form that if only one of the functions P and Q is assumed to be known it may be possible to determine the other function and thus satisfy (1). Since the constant of integration for the right member of (6), say, may be expressed as a logarithm with a negative sign it may then be used as a multiplier of the right member when Q is expressed as an exponential.

First consider the condition under which the right member of (5) is exact. The numerator then must be the exact derivative of the denominator or

$$2kQQ' - Q'\int Qdx - Q^2 = 2Q^2 - Q'\int Qdx, \text{ whence } \frac{Q'}{Q} = \frac{3}{2k} \text{ and } Q = e^{\frac{3x}{2k}}.$$

Substituting this value of Q in the right member of (5) and assuming the constant of integration of $\int Qdx$ to be zero, one finds $P = \frac{I}{3} Cke^{\frac{3x}{k}}$. These values of P and Q satisfy (1). Since P appears on both side of (1) the constant $\frac{1}{3}Ck$ may be divided out and need not appear in the expression for P.

Next consider the case when $P = Q$. Replacing Q and Q' by P and P' respectively in the first expression of (5), simplifying and integrating, one finds

$$P = Ce^{\frac{2x}{k}} = Q, \text{ whence (1) becomes } \left[\int e^{\frac{2x}{k}} dx\right]^2 = \int e^{\frac{4x}{k}} dx.$$

If one assumes special functions of x for the function Q, say, and substitutes these functions in (5), the integration generally becomes very involved, but may be carried out in some cases by assigning k definite values. A few illustrations of some interest will be given.

Assume $Q = x^m$. Substituting in (5) and simplifying one finds the integration simple and $P = C\left[k(m+1) - x\right]^{-(m+2)}$. However, the solution fails for $m = -1$. The substitution of these expressions for P and Q in (1) leads to the interesting relation

$$\int x^m dx \int \frac{dx}{[k(m+1) - x]^{m+2}} = \int \frac{x^m dx}{[k(m+1) - x]^{m+2}}.$$

Next assume $Q = e^x \cos x$. Here, upon substituting in (5), one finds that the integration may be readily effected if $k = \frac{1}{2}$ and the special value of P is then found to be $P = C\cot x\csc^2 x$. The generally solution with k arbitrary is very involved and was not carried out.

The substitution of a simple function for Q frequently leads to a complicated expression for P such that $\int Pdx$ becomes a difficult problem. Thus $Q = \cos mx$ where m is an arbitrary constant, gives upon setting $k = \frac{1}{m}$, $P = \frac{e^{\frac{mx}{2}}\cos mx}{(\cos mx - \sin mx)^{3/2}}$, from which it is evident that $\int Pdx$ is not easy of solution.

As was stated in the beginning this is a preliminary report and work is to be continued with the expectation or hope that the product of two integrals each of which may be integrated by the usual methods may give an integral that does not readily yield to known methods. Also it is hoped that more general properties of P and Q may be determined.