A Method of Visualizing Four Dimensional Rotations

JOHN KRONSBEIN, Evansville College

Three dimensional (XYZ) space with rectangular coordinates may be radically projected upon a unit sphere in 4 dimensional ($x_1x_2x_3x_4$) space by means of the formulae

$$(1)\quad x_1 = \frac{X}{[R^2+X^2+Y^2+Z^2]^{1/2}} \qquad x_2 = \frac{Y}{[R^2+X^2+Y^2+Z^2]^{1/2}}$$
$$x_3 = \frac{Z}{[R^2+X^2+Y^2+Z^2]^{1/2}} \qquad x_4 = \frac{R}{[R^2+X^2+Y^2+Z^2]^{1/2}}$$

The unit sphere S_4 in four-space may then be stereographically projected upon spherical 3-space with (xyz) coordinates by means of the formulae

$$(2)\quad x_1 = \frac{2Rx}{R^2+x^2+y^2+z^2} \qquad x_2 = \frac{2Ry}{R^2+x^2+y^2+z^2}$$
$$x_3 = \frac{2Rz}{R^2+x^2+y^2+z^2} \qquad x_4 = \frac{R^2-x^2-y^2-z^2}{R^2+x^2+y^2+z^2}$$

These formulae (2) are the generalization of the standard formulae for stereographic projection, and it is seen that the coordinates (xyz) represent a set of parameters for the unit sphere S_4.

Combination of (1) and (2) yields the relations

$$(3)\quad \frac{Rx}{X} = \frac{Ry}{Y} = \frac{Rz}{Z} = \frac{R^2-x^2-y^2-z^2}{2R} = \frac{R^2}{R+[R^2+X^2+Y^2+Z^2]^{1/2}}$$

and

$$(4)\quad x = \frac{X}{R+[R^2+X^2+Y^2+Z^2]^{1/2}}; \qquad y = \frac{Y}{R+[R^2+X^2+Y^2+Z^2]^{1/2}};$$
$$z = \frac{Z}{R+[R^2+X^2+Y^2+Z^2]^{1/2}}.$$

The rotations of the four-dimensional sphere S_4 constitute the group of (proper) quaternary transformations of the variables ($x_1x_2x_3x_4$) that are linear and homogenous and whose coefficient matrix $A=[a_{ik}]$ is orthogonal: $AA' = 1$, the dash meaning transposition of rows and columns of the matrix. The relations (1) show that they induce linear *fractional* transformations of the quantities (X/R, Y/R, Z/R) with coefficients which are components of the **corresponding quaternary** orthogonal matrix. These transformations are

$$\frac{X}{R}=\frac{a_{11}X'+a_{21}Y'+a_{31}Z'+a_{41}R}{a_{14}X'+a_{24}Y'+a_{34}Z'+a_{44}R}; \qquad \frac{Y}{R}=\frac{a_{12}X'+a_{22}Y'+a_{32}Z'+a_{42}R}{a_{14}X'+a_{24}Y'+a_{34}Z'+a_{44}R};$$

(5)

$$\frac{Z}{R}=\frac{a_{13}X'+a_{23}Y'+a_{33}Z'+a_{43}R}{a_{14}X'+a_{24}Y'+a_{34}Z'+a_{44}R}.$$

It would have been possible to use homogeneous coordinates instead of (XYZ); then the quaternary orthogonal transformations of S_4 would have induced the same transformations of these, but for a factor; the geometrical significance that follows below would, however have been less apparent.

The transformations (5) have the property that they leave invariant the quadric $R^2+X^2+Y^2+Z^2$ which may be regarded as fundamental quadric for the establishment of a metric in the (XYZ) space. Since this is a positive definite quadratic form, the metric must be elliptic. The (XYZ) space subject to the group of transformations (5) therefore has an elliptic metric impressed upon it.

The relations (2) show that the quaternary orthogonal transformations of $(x_1x_2x_3x_4)$ also induce a group of transformations in spherical (xyz) space. These are

$$(6) \qquad x = \frac{2a_{11}x'+2a_{21}y'+2a_{31}z'+a_{41}(R^2-x'^2-y'^2-z'^2)}{2a_{14}x'+2a_{24}y'+2a_{34}z'+(a_{44}+1)R^2+(1-a_{44})(x'^2+y'^2+z'^2)}$$

$$y = \ldots\ldots\ldots$$

$$z = \ldots\ldots\ldots$$

with similar expressions for y and z. They are clearly not linear, and in fact are easily shown to be inversions. The coefficient matrix A does not, however exhaust the inversion group in (xyz) space, since the fundamental invariant of the operations (6) is $R^2+x^2+y^2+z^2$. The group which leaves this expression unchanged is called the spherical group, a subgroup of the full (proper) inversion group that has the property of transforming socalled diametral points into similar points. Diametral points are pairs of points lying upon euclidean straight lines through, and separated by, a fixed point O, and so that the product of their (euclidean) distances from O is R^2.

The transition from the spherical (xyz) space to the elliptic (XYZ) space is by means of the formulae (4) which were obtained through the medium of the unit sphere S_4 in four space. The use of four dimensional space can however be avoided by projecting stereographically every plane through O upon a sphere of radius R with O as center; then moving the sphere at right angles to this plane by a distance R; then radically projecting the sphere back again upon the same plane. In this manner diametral points are brought to visible coincidence as shown in Fig. 1. The identification of diametral points represents the conversion of spherical to elliptic space, and it is this transition together with the use of special coordinates (XYZ) in the (xyz) space that permits the visualization of four dimensional rotations.

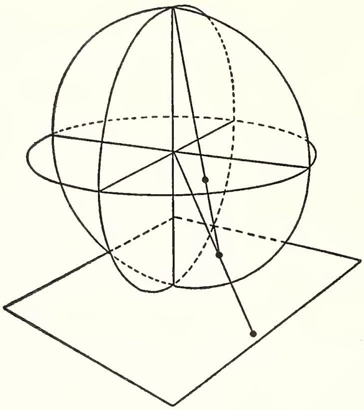

Fig. 1

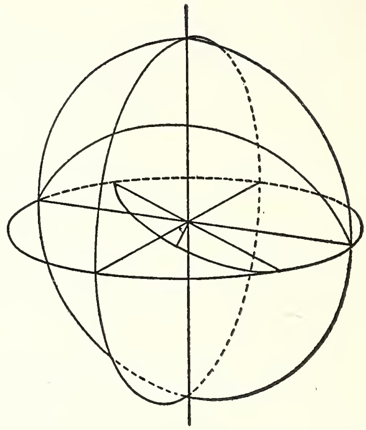

Fig. 2

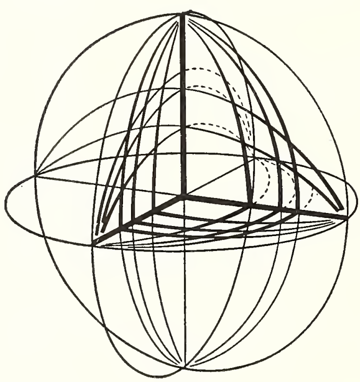

Fig. 4

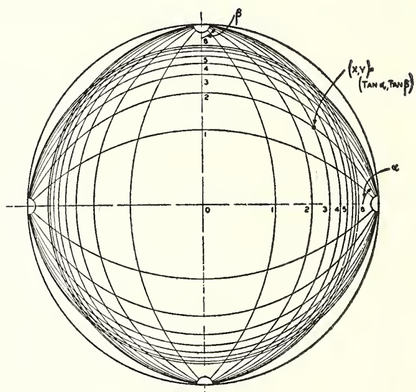

Fig. 5

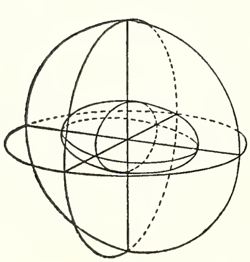

Fig. 6

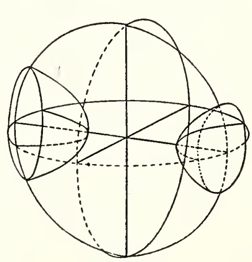

Fig. 7

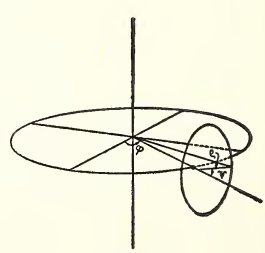

Fig. 8

Great circles on the unit sphere S_4 in 4-space are mapped by this procedure upon circles in (xyz) space through diametral points, and therefore also diametrically opposite points of the sphere S_R of radius R and center O in this spherical space, or upon euclidean straight lines through O. (The latter are special circles in (xyz) space). Fig. 2 shows that non-intersecting circles through diametrically opposite points of S_R are linked. As these are the images of non-intersecting great circles upon S_4, this property of the great circles on S_4 becomes intuitive.

Elliptic planes are mapped upon spheres through great circles on S_R, and the original (XYZ) coordinates which were rectangular, map upon a triple set of circles terminating in the ends of Cartesian coordinates axes on this sphere. Fig. 3 shows a plastic model of the (xyz) coordinate planes with the images of the original (XYZ) coordinates scribed upon them. Fig. 4 shows a perspective drawing of the

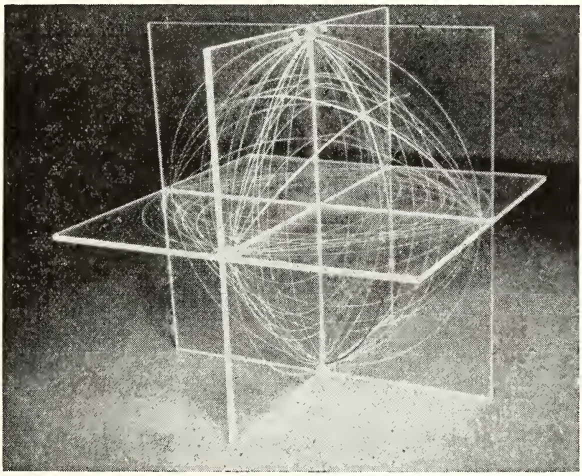

Fig. 3. Plastic model of the coordinate planes (See text).

same set of lines (circles) and Fig. 5 shows the new coordinates in the (XY) plane. The correlation between the numbers and the coordinate lines, by means of the tangent relationship given here indicates that the sphere S_R assumes the rôle of an "infinitely distant" plane. The great circles on S_R assume the rôle of "infinitely distant" straight lines on the "infinitely distant" plane. Now there are no such entities as "infinitely distant" lines points or planes in elliptic geometry so it will be preferable to denote these by the adjective "inaccessible' and thus speak of an "inaccessible line" lying in an "inaccessible plane". Actually, the elliptic distance between two points $(X_1Y_1Z_1)$ and $(X_2Y_2Z_2)$ is given by

$$d_{12} = \text{arc cos} \frac{R^2+X_1X_2+Y_1Y_2+Z_1Z_2}{[R^2+X_1^2+Y_1^2+Z_1^2]^{1/2}[R^2+X_2^2+Y_2^2+Z_2^2]^{1/2}}$$

so that the "inaccessible plane" has an elliptic distance of $\pi/2$ units of length from the origin, the point O.

The coordinate lines are seen to meet in a point on the inaccessible plane and thus give the impression of being parallel lines similar to what is considered parallel in the euclidean space. But it should be observed that they do not possess the property of euclidean parallels that the intercepts on the set of common normals are of equal length. Upon expansion of S_R so that $R \longrightarrow \infty$, they degenerate into euclidean parallels, and therefore will be referred to as "parallels of the first kind". The reason for the term "first kind" will appear below.

There are only two shapes of (real) quadrics in elliptic space, the hyperboloid of two sheets and the ellipsoid being essentially identical. Algebraically this follows from the inertia property of quadratic forms, but in the present model of the elliptic space it is easy to see the reason. In Figs. 6 and 7 are seen respectively the ellipsoid and the hyperboloid of two sheets. Owing to the identity of diametral points the ellipsoid in Fig. 6 must comprise another circuit outside S_R and inverse with respect to it. Now effect a transformation of the spherical group which transforms the (euclidean) plane that halves the ellipsoid into S_R The left half of the ellipsoid then becomes the right hand sheet of the hyperboloid in Fig. 7. The other sheet is obtained from the outer circuit of the ellipsoid. In two dimensions, bicircular quartics behave in analogous manner (2), and assume the rôle of (sphero-) conics there. Algebraically, these shapes correspond to the following quadratic forms reduced to principal axes:

$$X^2+Y^2+Z^2+R^2=0 \quad \text{(imaginary quadric)}$$
$$X^2+Y^2+Z^2-R^2=0 \quad \text{(oval quadric)}$$
$$X^2+Y^2-Z^2-R^2=0 \quad \text{(ring shaped quadric).}$$

That the second and third quadrics should differ from one another is obvious on grounds of connectivity alone. In euclidean space these connections between the different configurations are not so easily surveyed.

If the transverse generating circles of a torus intersect S_R at right angles, the torus represents a hyperboloid of revolution of one sheet. Fig. 8 shows one transverse generating circle being taken around the equator circle of S_R. The equation of the torus is seen to be

$$(7) \qquad x = (a+\rho\cos\psi)\cos\phi;\ y=(a+\rho\cos\psi)\sin\phi;\quad z=\rho\sin\psi,$$

where $a^2=R^2+\rho^2$. Elimination of ψ and ϕ yields

$$\frac{\rho^2 \quad (4x^2+4y^2)}{[\,R^2-x^2-y^2-z^2\,]^2} - \frac{4R^2z^2}{[\,R^2-x^2-y^2-z^2\,]^2} = 1$$

By formulae (2) this immediately transforms to

$$(8) \qquad \frac{X^2+Y^2}{R^4/\rho^2} - \frac{Z^2}{R^2} = 1,$$

which proves the statement. The quadric (8) in elliptic space is a quartic surface in inversion space (spherical space). The reguli of this so-called *Clifford* surface are elliptic straight lines, euclidean circles in inversion space, oblique generator circles of the torus illustrated in

Fig. 9 and the plastic model shown. There is a *left* and a *right* regulus, but only one is shown. Which of the two is denoted by left and which by right, is of course immaterial.

The entire elliptic space can be filled with Clifford surfaces with a common generator axis, corresponding to increasing radii of transverse generator circles. By consideration of individual sets of reguli it is seen that the surfaces can be generated by translation of points along either "right" or "left" regulus lines. During such a translation a "right" slides along a "left" regulus, or a "left" along a "right" regulus to generate the torus. Such a movement is called a "Clifford Translation". All points in the elliptic space move along elliptic straight lines during such a translation.

In euclidean space any two non-intersecting straight lines have either one common normal or an infinite number of them, and in the latter case the straight lines are called "parallel". In elliptic space two elliptic straight lines have either *two* common normals or an infinite number of them, and in the latter case the intercepts of the normals between them are of equal elliptic length. This means that two such lines are equidistant in elliptic measure, and that is exactly the distinctive property of the reguli of a Clifford surface. Such lines are called "Clifford Parallels" in view of the resemblance to euclidean parallels they bear in respect of distance. Eduard Study (3) called them "paratactics" and distinguished "right" and "left" paratactics, according as two such lines belonged to one or the other set of reguli of a hyperboloid of one sheet (the lines would of course be suitably oriented—reference to this will be made below). The designation "paratactics" is more suitable than "parallels" because upon indefinite expansion of S_R so that $R \longrightarrow \infty$, the parallels of the first kind described above become euclidean parallels, but the Clifford Parallels do not. It is appropriate, therefore to name the Clifford parallels "of the second kind".

Given any two elliptic straight lines, L_1, L_2, it is possible to find a transformation of the elliptic space (and hence also of the spherical model) which carries the intersection of one of them, L_2, say, and one of the common normals of L_1, L_2, into the fixed point O and at the same time to transform the elliptic plane determined by the normal and L_2 into a euclidean plane. L_2 will then be a euclidean straight line (as well as elliptic), and the same applies to the common normal. L_1 will be represented by a circle through diametrically opposite points of S_R Fig. 10 shows this configuration in such a position that the common normal (of L_1 and L_2) is transformed into the z-axis (which is also the Z-axis) while L_2 is a euclidean straight line passing through diametrically opposite points of the equator of S_R (which represents an inaccessible line). If in this configuration the angle $\alpha=\beta$, the two lines L_1 and L_2 are Clifford parallels, and they will then have an infinite number of common normals, the elliptic length of the intercepts between them having the common length α. It will be noticed that unless two straight lines are Clifford parallels, the intercepts on the two common normals are not equal in elliptic length.

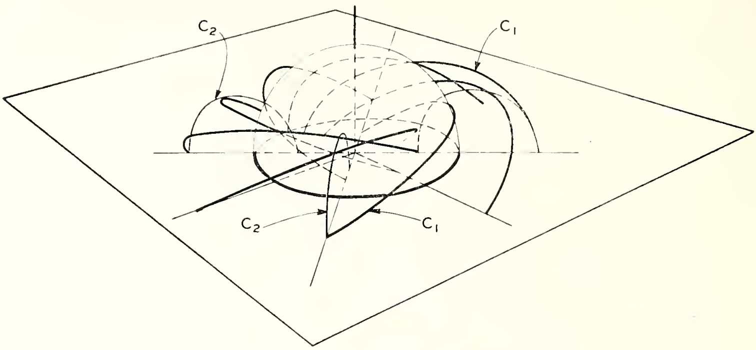

Fig. 9

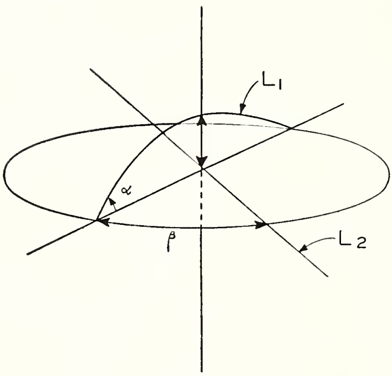

Fig. 10

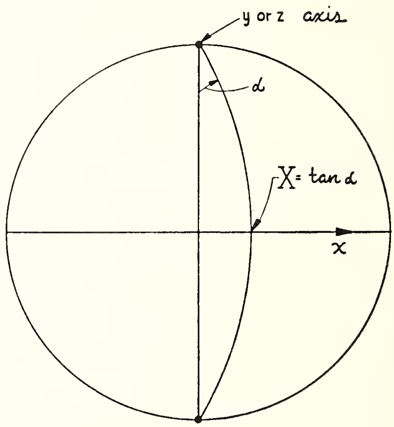

Fig. 11

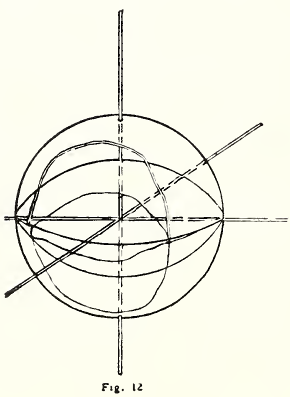

Fig. 12

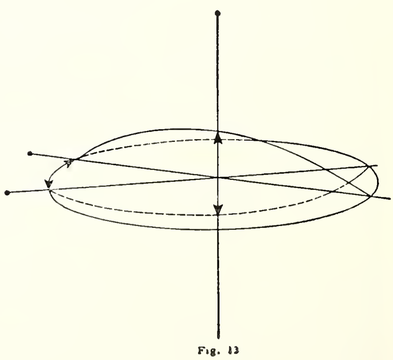

Fig. 13

Another example of a pair of Clifford parallels visible in Fig. 10 is the limiting case where $\alpha=\pi/2$, represented e.g. by the Z-axis and the equator circle. Such a pair of special paratactics or Clifford Parallels will be called a *Study Line cross*, being a generalization of what Study called a "line cross" (3) namely, a straight line in the finite part of (euclidean) space and that inaccessible (infinitely distant) line which crosses it at right angles. In our case a general Study Line Cross will be represented by a pair of circles lying in (euclidean) planes orthogonal to each other and having one (and consequently infinitely many) shortest distance of length $\pi/2$. The line cross will be used in the development of the kinematics below.

If in the transformation formulae (6) the following special coefficient values are inserted:

$a_{11}=a_{44}=\cos\alpha$; $a_{14}=-a_{41}=\sin\alpha$; $a_{22}=a_{33}=1$, and all other coefficients zero, the matrix $|a_{ik}|$ is seen to be still orthogonal, and (6) becomes (8)

$$\begin{aligned} x' &= [2\cos\alpha\, x + \sin\alpha\, (R^2 - x^2 - y^2 - z^2)]/N \\ y' &= 2y/N \\ z' &= 2z/N, \end{aligned}$$

where

$$N = -2\sin\alpha\, x + R^2\ (1+\cos\alpha) + (1-\cos\alpha)\ (x^2+y^2+z^2).$$

This transformation will be called a "pseudorotation" about a circle of radius R lying in the (yz) plane, when considered in the spherical or inversion space. It leaves invariant the circle $y^2+z^2=R^2$ in the (yz) plane, which represents an inaccessible line in the inaccessible plane S_R in the elliptic space. It will be seen from Fig. 11 that the plane determined by the y and z-axes has been rotated through an angle α, and this transformation may be generated by a reflection (inversion) in a sphere making an angle ½ α with the (yz) plane, followed by a reflection in the origin, i.e. $x'=-x$, $y'=-y$, $z'=-z$.

The rotations of 4-space now become visible as kinematics in elliptic (XYZ) space or spherical (xyz) space, in either case representable by our spherical model illustrated in Fig. 3. The transformation (8) can be illustrated by Fig. 12 which shows how a euclidean plane through O is transformed into a spherical shell through the equator circle (elliptic plane passing through inaccessible line represented by equator circle.) The 6 "simple" rotations of Cartan's "biplan" are seen to be on the one hand, euclidean rotations of the planes (xy), (xz) and (yz), and on the other the elliptic rotations about inaccessible lines represented by circles of radius R in the (yz), (xz) and (xy) planes, respectively. The last three are obviously pseudorotations in the sense defined above. These relations are shown in Table I, which shows the quaternary orthogonal matrices corresponding to these in the second column, and the geometrical significance in the fourth and fifth columns.

Fig. 13 shows the mechanism of a rotation as well as of a pseudorotation.

The third column in the table shows what becomes of the matrices in the second for angle of rotation or pseudorotation of $\pi/2$. These matrices are commutative with respect to multiplication and certain pairs of them multiplied together yield the 4 x 4 matrices used by Gibbs to represent quaternion units (1). Thus

TABLE I. The Six Simple Quaternary Orthogonal Matrices.

Matrix Symbol	Matrix	Corresponding 90° Rotation	Significance in 4-space	Significance in elliptic or inversion 3-space
M_{12}	$\begin{matrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$	Rotation in (x_1x_2) plane	Elliptic *and* euclidean Rotation in (xy)-plane
M_{13}	$\begin{matrix} \cos\alpha & 0 & -\sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$	Rotation in (x_1x_3) plane	Elliptic *and* euclidean Rotation in (xz)-plane
M_{23}	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$	Rotation in (x_2x_3) plane	Elliptic *and* euclidean Rotation in (xz)-plane
M_{14}	$\begin{matrix} \cos\alpha & 0 & 0 & -\sin\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\alpha & 0 & 0 & \cos\alpha \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +1 & 0 & 0 & 0 \end{matrix}$	Rotation in (x_1x_4) plane	Elliptic Rotation *about* inaccessible line in inaccessible plane. Line represented by unit circle in (yz) plane.
M_{24}	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & -\sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & \sin\alpha & 0 & \cos\alpha \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & +1 & 0 & 0 \end{matrix}$	Rotation in (x_2x_4) plane	Elliptic Rotation *about* inaccessible line in inaccessible plane. Line represented by unit circle in (xz) plane.
M_{34}	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{matrix}$	Rotation in (x_3x_4) plane	Elliptic Rotation *about* inaccessible line in inaccessible plane. Line represented by unit circle in (xy) plane.

$$M_{12}(\pi/2) \times M_{34}(\pi/2) = i = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

$$M_{13}(\pi/2) \times M_{24}(\pi/2) = j = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}$$

$$M_{23}(\pi/2) \times M_{14}(\pi/2) = k = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

The most general transformation of XYZ space is one which carries an elliptic straight line into another one. The first may be arbitrarily chosen and the latter then preassigned. It is possible to lay an elliptic plane through the given line and one of the common normals, and then to carry out a pseudorotation about that inaccessible line which joins points where the given line and the common normal chosen pierce the inaccessible plane. This pseudorotation is carried out so that the line and the normal have their intersection in the fixed point 0 after the operation. These two lines are then both euclidean and elliptic and they therefore appear as in Fig. 10, and they will be Clifford parallels if $\alpha = \beta$. The given line L_1 may now be moved into the other L_2 by first carrying out a euclidean rotation through the angle β, and following this by an elliptic rotation through an angle α about the inaccessible line crossing the axis of the euclidean rotation at right angles. This pair of operations, i.e. rotations carried out successively about two lines

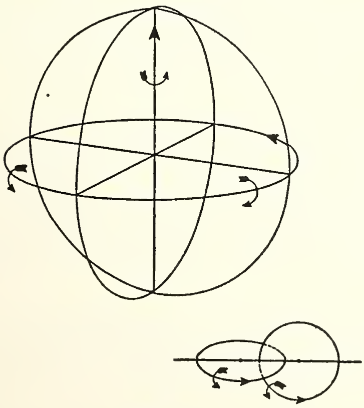

Fig. 14

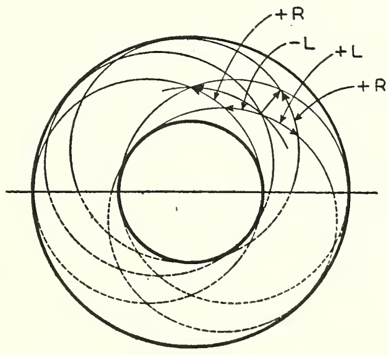

Fig. 15

of a Study Line Cross is commutative: the elliptic pseudorotation about the inaccessible line could have been carried out first, and this followed by the euclidean rotation about the euclidean axis.. Thus, but for a transformation of the spherical group placing the two lines into a special position in the space, the given line is moved into the other by a euclidean rotation about a euclidean axis, followed by a pseudorotation about an inaccessible axis orthogonal to the first. Or, also, *the most general movement in elliptic space consists of the succession of two (elliptic) rotations whose axes form a Study Line Cross.*

It will be obvious that the orthogonal rotations in four-space corresponding to these operations in elliptic or spherical space will decompose in an analogous manner.

The pseudorotation leaves invariant a single infinity of toruses namely those whose common generator axis is the axis orthogonal to the pseudorotation. Individual transverse generator circles of each torus remain invariant as a whole. Thus, this movement is identical with that of a smokering revolving within itself, and is frequently called a "vortex motion".

The decomposition of the quaternary rotations into two, about skew circles on S_4 is not the most elementary one. Each rotation may be seen to be composed of two Clifford translations. It is not difficult to show that the straight lines of the elliptic space may be oriented so that the (arbitrarily chosen) *right* regulus together with the common axis of rotation form a right handed screw in the conventional sense, and the *left* will then be *eo ipso* oriented so that it together with the same axis of rotation, whose orientation remains unchanged, form a left hand screw. Diagrammatically this is illustrated in Fig. 15 where the common axis of the hyperboloids is thought of as directed out of the plane of the paper towards the observer, and a positive right as well as a positive left, generator direction are indicated. The generator circles are imagined collapsed into the plane as circles.

It is now immediately visible that a euclidean rotation about the common axis of composed of a *negative left* Clifford translation, followed by a *positive right* Clifford translation. Similarly, it is seen that a pseudorotation is composed of a *positive left* followed by a *positive right* Clifford translation.

All the preceding geometrical relations, interesting as they are, become increasingly signficant in view of the last statement. For, it is now possible to correlate without trouble the kinematics in elliptic or spherical space with the algebraic expressions for quaternary orthogonal transformations. It is well known that the general (proper) quaternary orthogonal transformations may be represented by the quaternion formula

(9) $|A|\ |B|X'=\overline{A}XB$, where the A, B, and X are quaternions:

$$A=ia_1+ja_2+ka_3+a_4;\quad |A|=[a_1^2+a_2^2+a_3^2+a_4^2]^{1/2}$$

and smilarly for B, X, and X′. $\overline{A}$ is the conjugate quaternion, obtained from A by using —i, —j, —k instead of the positive units.

It is also known that with this representation, euclidean rotations may be expressed as

(10) $|A|^2X'=\overline{A}XA$

and the expanded expressions then form the *Euler-Rodriques-Cayley* parametric representation of the ternary orthogonal matric elements. These are shown in table 2, where it should be observed that

$$a_1^2+a_2^2+a_3^2+a_4^2$$

is the denominator of the other nine elements when the matrix is used to express the ternary orthogonal transformations.

It is then not difficult to show that the expression

(11) $|A|^2X'=AXA$

is a pseudorotation about the inaccessible line that crosses the axis of the euclidean rotation $\bar{A}XA$ orthogonally, i.e. the second elliptic line of a Study line cross. The angle of this pseudorotation equals that of the corresponding euclidean rotation. These statements may be proved in two ways, either by direct calculation, or by first transforming the inaccessible line which forms the axis of the pseudorotation into the Z-axis. This converts the pseudorotation into a euclidean rotation about the Z-axis, and the proof is then obvious.

Further, one can then *see* that a transformation such as $|A|X'=AX$ may be correlated to a positive left translation, while $|A|X'=\bar{A}X$ then represents a negative left translation, both in the Clifford sense. Simi-

TABLE II. Parametric Representation of Rotations and Pseudorotations

General Quaternary Rotation $|A|\,|B|X'=\bar{A}XB$; $A=ia_1+ja_2+ka_3+a_4$
Euclidean Rotation $B = A$: $|A|^2X' = \bar{A}XA$
Euler-Cayley-Rodrigues Coefficient Matrix:

$$\begin{array}{cccc}
a_1^2-a_2^2-a_3^2+a_4^2 & 2(a_1a_2-a_3a_4) & 2(a_1a_3+a_2a_4) & 0 \\
2(a_1a_2+a_3a_4) & -a_1^2+a_2^2-a_3^2+a_4^2 & 2(a_2a_3-a_1a_4) & 0 \\
2(a_1a_3-a_2a_4) & 2(a_2a_3+a_1a_4) & -a_1^2-a_2^2+a_3^2+a_4^2 & 0 \\
0 & 0 & 0 & a_1^2+a_2^2+a_3^2+a_4^2
\end{array}$$

Pseudorotation $|A|^2X' = AXA$. Axis of this is inaccessible line crossing orthogonally axis of above Rotation.

$$\begin{array}{cccc}
-a_1^2+a_2^2+a_3^2+a_4^2 & -2a_1a_2 & -2a_1a_3 & -2a_1a_4 \\
-2a_1a_2 & a_1^2-a_2^2+a_3^2+a_4^2 & -2a_2a_3 & -2a_2a_4 \\
-2a_1a_3 & -2a_2a_3 & a_1^2+a_2^2-a_3^2+a_4^2 & -2a_3a_4 \\
+2a_1a_4 & +2a_2a_4 & +2a_3a_4 & -a_1^2-a_2^2-a_3^2+a_4^2
\end{array}$$

TABLE III. Matrix of Coefficients for General Quaternary Rotation.

$|A|\ |B|X' = \overline{A}XB$ in quaternion notation

$a_1b_1 - a_2b_2 - a_3b_3 + a_4b_4$	$a_1b_2 + a_2b_1 - a_3b_4 - a_4b_3$	$a_1b_3 - a_2b_4 + a_3b_1 - a_4b_2$	$a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1$
$a_1b_3 + a_2b_4 + a_3b_1 + a_4b_2$	$a_1b_4 - a_2b_3 + a_3b_2 - a_4b_1$	$-a_1b_1 - a_2b_2 + a_3b_3 + a_4b_4$	$-a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3$
$a_1b_2 + a_2b_1 + a_3b_4 + a_4b_3$	$-a_1b_1 + a_2b_2 - a_3b_3 + a_4b_4$	$-a_1b_4 - a_2b_3 + a_3b_2 + a_4b_1$	$a_1b_3 - a_2b_4 - a_3b_1 + a_4b_2$
$-a_1b_4 + a_2b_3 + a_3b_2 - a_4b_1$	$a_1b_3 + a_2b_4 - a_3b_1 - a_4b_2$	$-a_1b_2 + a_2b_1 - a_3b_4 + a_4b_3$	$a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$

larly, $|A|X'=X\overline{A}$ is a negative right translation, and $|A|X'=XA$ is a positive right translation.

The coefficient matrix for a pseudorotation is shown in the lower part of table 2; it is the parametric representation of a vortex motion. When these parameters are inserted in (6), the motion is in inversion space. The geometrical meaning is also easy to deduce, since the significance of the Euler-Rodrigues-Cayley parameters is well known. Thus, $a_1:a_2:a_3$ is known to represent the ratio of direction cosines of the rotation (or pseudorotation) axis, while $-a_4/[a_1^2+a_2^2+a_3^2]^{1/2}$ cotangent of half the angle of rotation.

Table 3 shows the coefficient matrix of a general quaternary orthogonal transformation. When the radius of S_R is made to increase indefinitely, there results a parametric representation of *all* euclidean movements, i.e. rotations as well as translations.

A difficulty arises here, however, inasmuch as the representation is in terms of 8 homogeneous parameters with bilinear composition. The totality of euclidean movements only depends upon 6 essential constants, so that we have one excess constant.

Eduard Study (2) showed that there is no representation in terms of 7 homogeneous parameters with bilinear composition for the totality of euclidean movements, and he assumed that the expression $a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 = 0$. Study deduced his result by the aid of Clifford dual numbers, i.e. complex numbers $a + eb$ for which a and b are real numbers while e is a unit for which $e^2 = 0$. The totality of euclidean movements is then expressible in the following form using Biquaternions:

$$(a+eb)^{-1}(e[ix_1+jx_2+kx_3] + x_4)\ (a-eb) =$$
$$e[ix'_1+jx'_2+kx'_3] + x'_4\ .$$

This representation is complete and exhaustive. It is seen that our limiting process $R \longrightarrow \infty$ causes degeneration of pseudorotations into euclidean translations.

Literature Cited

1. Commentary on the collected works of J. WILLARD GIBBS, Vol. II.
2. KRONSBEIN, J. 1946. Elliptic geometry, conformal maps, and orthogonal matrices. Duke Math. Jour. 13:505-519.
3. STUDY, E. 1903. Geometrie der Dynamen. Leipzig.