## PRESIDENTIAL ADDRESS

# The Role of Mathematics in Science

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When mathematicians are assembled to hear a talk by one of their own kind it is expected that the speaker define his terms. If there be terms that cannot be defined they should be listed. It is generally accepted that mathematics begins with a set of postulates, definitions, and undefined terms. Consequently I should define the terms mathematics and science. A formal definition of either would be unsatisfactory. Each of you is engaged in some phase of scientific work and any definition which I might give of science most surely would fail to satisfy some of you. I list it as an undefined term, hoping that in a general way we hold like views as to the broad meaning of the term.

Mathematics has been defined in many ways. Bertrand Russell in his Philosophy of Mathematics defined mathematics as ". . . the subject in which we never know what we are talking about, nor whether what we are saying is true." While the statement has brought forth many different reactions, I will let it stand without comment for the present. I shall use the term more in the common sense of a system of symbols and laws of operation. This has been expressed by the statement that mathematics is the science of drawing necessary conclusions.

While not attempting to define the term science I believe we can agree that in the development of the various fields of science four phases are recognizable. First is the observation and recording of facts. Second is the formation of theories based on generalizations of the observations. Third is the drawing of necessary conclusions which must follow if the statements of theories represent basic laws. Fourth is the testing of our conclusions by controlled experiments.

Recognizing the importance of each part to the development of the whole I wish to avoid giving the impression that I regard one part as more scientific than another. A good example of the observation and recording of facts is the work of the State Flora Committee. The publication of the work of this committee in the Proceedings has placed the Indiana Academy of Science in a position of leadership among all state academies in this field. The work on the fauna of Indiana is less complete but important contributions have been published by the Entomologists.

The biological sciences are not the only ones in which the collection and recording of data is a prominent part of the work. Perhaps the largest organization of observers is in the field of meteorology. The importance of the coordination of workers in this field came to public attention during World War II. World wide organizations of observers also exist in Seismology and in the oldest of all sciences Astronomy.

To illustrate the second phase I turn to the field of Astronomy. From the dawn of history men have observed the stars and the motions of planets, moon, and sun. Many theories concerning the celestial bodies were formulated. About 150 A. D. Ptolemy developed his geocentric theory which influenced the thinking of scientists for more than a thousand years. It was a theory of structure, attempting to explain how the bodies moved. It was not a theory which explained the motions on the basis of an underlying physical law. The same can be said of the theory of Copernicus which was published shortly before his death in 1543. His theory was less complicated from the standpoint of the mathematics involved but it, too, was only a theory of how the motions were related. The classic example of the organization of data of observation is the work of Kepler (1571-1630) who spent the major portion of his adult life studying the records of observation left by the greatest observer of all time, Tycho Brahe. Out of a lifetime of work Kepler gave the world three results which today are known as Kepler's Laws. The first stated that the orbits of the planets were ellipses with the sun at a focus. The second stated that the planets moved at varying speeds so that the area swept over by a line joining sun to planet would be proportional to the time. The third stated that the cubes of the distances of any two planets from the sun are to each other as the squares of their periodic times. Kepler seems to have taken especial delight in the third law. Of it he wrote: "The die is cast, the book is written, to be read either now or by posterity, I care not which; it can await its reader; has not God waited six thousand years for an observer?"

It is often said that without the work of Appolonius on Conics, there would have been no Kepler. Like Ptolemy and Copernicus, Kepler attempted to explain how the planets move. He did not attempt to state the physical law that gave the cause of their movement. With Newton the science of Astronomy entered the third phase. Beginning with Newton's statement of the universal law of gravitation and using the methods of Calculus it is possible to contain Kepler's laws not in twenty years but in less than two hours.

The third phase of science which I listed was the drawing of necessary conclusions. It is in this phase that mathematics plays its most important part. Some would even identify this process with the very definition of mathematics. However the work of Kepler was also mathematical in character but based on a portion of mathematical theory which was not adequate to perform the task accomplished by Newton. To this phase of science and especially the work of Newton I will return.

The fourth phase of the development of a science is the testing of predictions by experiment. In the field of Botany, the geneticists have given mid-west America hybrid corn. This is not the result of blind experiment but of planned experiments based on theories concerning the fundamental laws of inheritance. In the field of Chemistry we have arrived at the point where in many cases it is possible to predict the properties of a chemical compound before the compound has been synthesized. Progress has been made in this line in drugs, plastics, and synthetic fibers.

The worker needs to continue to be an observer of facts. Important discoveries continue to be made by the study of experiments which give unexpected results. Today when we ride in our automobiles we are protected by safety glass. The basic discovery which led to its invention was not the result of a planned experiment but instead was the result of an accident. It may be taken as a fundamental axiom of all science that if our basic assumptions include a true statement of all the physical laws involved and if our reasoning is valid, then there can be no deviation between actual and predicted result. The surprise result can occur only when there is an error in our reasoning or where there are factors present which were not included in our statement of conditions. The latter is the more frequent case. Expressed mathematically, we have omitted terms from our equations which are not negligible.

Returning to the definition of mathematics given by Russell we find that it makes sense if we regard mathematics as restricted to the mechanics of drawing conclusions from a set of symbols. The differential equation $dx/dt = kx$ expresses the law that the rate of change of a quantity x is proportional to the amount of the quantity x. The solution may be found by the sophomore student of Calculus without any reference to what sort of quantity is represented by x. In fact he need not know that the law is ever true. His conclusion can be stated that if the law expressed by the differential equation is ever true then the value of x at any time is given by the law $x = Ce^{kt}$ where C is the value of x when $t = 0$. The Actuary calls this the compound interest law. The Biologist calls it the law of growth. The same law arises in other fields and is known by other names. The mathematician performs an operation on a set of symbols. He need not know the meaning of the symbols, nor whether the condition stated is ever true. Of one thing he is certain. If the first equation is true the second must also be.

Basically pure mathematics is abstract. We humans share a common trait of thinking in concrete situations. We want meaning in our symbols, but the full power of mathematical reasoning can be realized only to the extent that we are able to substitute symbols for ideas and then draw valid conclusions by performing operations on the symbols. The concept of a one to one correspondence between classes is fundamental. The words of our number system are perhaps the first step in the long road of successive abstractions that each of us must travel in our mastery of mathematics. A child learns that two books added to three books makes a total of five books, that two apples added to three apples makes a total of five apples, etc. The idea that two plus three equals five is abstract. Each of us passed through a stage at which we could add small sums if each bore a label but found the process difficult when the sums were not so labeled.

It is often stated that the greatest invention of man was the symbol for the number zero. It made possible the arabic system of numerals. With the arabic numerals it was possible to construct simple algorithms whereby the four fundamental operations could be performed directly on the symbols without the use of objects. To appreciate the meaning of this try to extract the cube root of 3 correct to four significant places using the Roman numerals. It is possible but not easy. It is interesting to speculate on the possible attainments of such mental giants as Archimedes had they possessed our present system of writing numbers. This is also the secret of the power of algebraic symbols.

The development of mathematics has been closely linked with advances in the sciences of Physics and Astronomy. Neither science has ever advanced beyond the stage of the available mathematical theory. In some cases the needed mathematics has been invented by the scientist in order to proceed with his scientific studies. The outstanding example is of course Sir Isaac Newton who was interested first of all in Physics and Astronomy. Had he not invented a new and powerful tool his scientific work could not have gone much beyond his notable predecessors: Ptolemy, Copernicus, and Kepler. Symbols could readily be written for forces, accelerations, velocities, and coordinates of position. The thing needed was a set of operations which could be performed on the symbols in an equation which would give other symbols which in turn could be interpreted as physical states. Newton's Fluxions accomplished this purpose.

The two operations which he defined are known as differentiation and integration. With these operations and three basic postulates which we know as Newton's Laws of Motion, plus his Universal Law of Gravitation the whole of classical mechanics followed. At approximately the same time, Liebnitz in Germany was working in the field of Geometry. Needing more general methods of finding tangents than were available in the Geometry of Descartes, he, independently of Newton, invented the same two transformations. The work of the two men differed in notation and in purpose but the resulting abstract mathematical operations were identical. The importance of this feature cannot be overemphasized. The laws of operation on symbols are independent of the subjective content of the symbols. It comes very close to the heart of mathematics and explains its power. Recall that in 1700 virtually nothing was known of electricity and the foundations of modern chemistry had not been formulated. Yet this abstract theory for the manipulation of symbols contains the most useful tools for the investigation today of problems in electrical theory and in Physical Chemistry.

Following Newton and Leibnitz there was a period of more than a century during which the pure mathematicians flourished and the Calculus expanded far beyond the needs of any scientist working in other fields. In most areas researches in pure mathematics kept well ahead of the applications. Foundations were laid for new and specialized fields. It was during this period that the foundations of Differential Equations, Calculus of Variations, Infinite Series, and other divisions of Analysis were laid by men from Euler to Weierstrass who were interested in abstract mathematical ideas. There were some notable exceptions. Fourier was first of all a Physicist. The series known by his name was invented to assist in the development of his theory of heat. Today Fourier series are of even more importance to the electrical engineer than to the heating engineer.

In 1871 when Clerk Maxwell published his treatise on Electricity and Magnetism the mathematical tools were available for him to write down his famous Field Equations. These appeared to be based on sound postulates. However the equations seemed to give too much. Solutions appeared for the equations that were not matched by any known physical phemenan. The story of Hertz and his discovery of wireless waves is known to you. Today radio and television are a part of our everyday life. Whether or not the technical developments could have been made without the beginning in a set of differential equations is uncertain. At least the fact remains that the equations were written and solved first and that Hertz's basic discovery was inspired by a set of equations.

In our own generation another mathematical equation has played a vital role in the making of history. I refer to the equation $E = mc^2$ which was first deduced by Einstein. It was known and accepted by mathematicians and physicists for a quarter of a century before the first successful attempt was made to release energy by the destruction of matter.

These two examples serve to emphasize the point made by Benjamin Pierce when he defined mathematics as the science of drawing necessary conclusions. One might add that at least in the case of Maxwell the definition of Bertrand Russell applies. Certainly when Maxwell wrote his equations he did not have radio and television in mind. Another example of the drawing of necessary conclusions by mathematical analysis is the discovery of the planet Neptune. The Englishman Adams and the Frenchman Leverrier each computed the orbit of an hypothetical planet which was assumed to exist and be the cause of the discrepancy between the actual and theoretical orbit of Uranus. Adams completed his work first but was unable to interest any English astronomers in seeking the planet. Leverrier was more fortunate. Neptune was first recognized by the German astronomer Galle who found the planet in less than half an hour within less than half a degree of the position predicted by Leverrier.

There have been periods in the history of science when the advances in science seem to overtake the inventions of the mathematicians. The period of Newton was one. Today problems arise in radar, television and other high frequency circuits which give rise to non-linear differential equations. The general theory is extremely difficult and until recently the equations were not considered either important or interesting. Today they are important and particular equations are being solved, often by methods of numerical approximation. Perhaps more general methods of solution will be found. At any rate the present lack of satisfactory general solutions emphasizes the close connection between physical and mathematical theory. It has been suggested by at least one competent physicist that the basic equations of physics must be non-linear, and that mathematical physics will have to be done over again. Should that prove true the analysts of the next century will face harder problems than have been solved to date and the result may be a system of mathematics quite different from that known today.

Most of you have some acquaintance with the elementary theory of vectors. A few of you may be familiar with some of the earlier theories of multiple algebras. Hamilton's quarternions are perhaps the best known of the early attempts to produce a system of this type. The vector analysis developed by J. W. Gibbs in the 1880's is still the most important of the simpler types. The development from vector to matrix and tensor was rapid and by 1900 both theories were well developed. However up until the second world war neither theory had much practical value to the engineer. As a graduate student I learned the theory of matrices.

But I also learned the difficulty of applying that theory. The calculation of the inverse of a square matrix of order two or three was fun. To calculate the inverse of one of order ten or twenty appeared impossible because of the time involved. The modern electronic digital computer has changed all that and matrices have become one of the working tools of engineering research. Much of the progress made in the development of jet planes that fly faster than the speed of sound would have been impossible without the mathematical theory of matrices.

As has happened so often in the history of the development of mathematics, matrices are proving of equal value in totally different fields. Today there are a number of electronic computers in use, each staffed by a working crew of from ten to thirty men, including mathematicians, electrical engineers, and technicians. At the meeting of the Econometric Society last September it was reported that the inverse of a square matrix of order 190 had been computed in an elapsed time of only 47 hours. This problem had arisen not in physical science but in economics. Had the problem been attempted by the methods discussed in courses in elementary theory of equations a little calculation soon reveals that the number of sheets of paper needed would far exceed Edington's estimate of the number of electrons in the known universe.

The present importance of computational methods gives emphasis to the statement made more than a century ago by the great German mathematician Gauss: "Mathematics is Queen of the Sciences and Arithmetic is the Queen of Mathematics. She often condescends to render service to astronomy and other natural sciences, but under all circumstances the first place is her due."

From the time of Euclid to the early part of the seventeenth century Geometry was regarded as an example of a complete and perfect mathematical theory. The work of Appolonius of Perga gave us all the properties of the conic sections almost seventeen centuries before they were applied to the theory of ballistics and through astronomy to the practical art of navigation. Then came the analytic geometry of Descartes, the Projective Geometry of Pascal and Desargue, the Descriptive Geometry of Monge which for a time was classified as a military secret by the French. Nor was the plane geometry of Euclid to continue to appear as a finished product. The number of new theorems stated by Steiner alone far outnumber the list in Euclid's Elements.

Perhaps the most important invention in the field of Geometry was the invention of the Hyperbolic Geometry by the Russian, Lobatschewski, and the Austrian, Bolyai. Paradoxically the system has little practical value. It is true that the addition of vectors in the hyperbolic plane can be made to yield the Fitzgerald contraction formula which is useful in Relativity. The importance of the invention lies in the effect it has had on the development of Geometry. It was a challenge to the accepted way of thinking to be forced to admit the existence of two self-consistent but mutually contradictory sets of theorems. Geometry ceased to be a branch of physical science and became a part of abstract logic. The latter part of the nineteenth century and the first half of the twentieth has seen the development of much that today exists as so-called pure mathematics. Differential Geometry, Algebraic Geometry, Analytic Projective Geometry, and Topology all show healthy active growth. Yet there is a strong suspicion that like the pure theory of matrices of 1900 these too may become the applied mathematics of the next half century.

In conclusion let us ask again what is the role of mathematics in science. One answer could be given by listing the various subdivisions of mathematics that are used by scientific workers in various fields. The list would be long, even for some of the sciences which we ordinarily think of as non-mathematical. For other sciences the list would be almost a complete roster of the subdivisions of mathematics. Instead of answering in this way I prefer to ask what there is about mathematics that gives the subject its peculiar power. I believe that the answer may be found at least in part in two fundamental concepts. The first is the concept of a one to one correspondence between a set of abstract symbols and the set of elements which form the body of subject matter of a science. The second concept is that which the mathematician calls a group. The group is defined as a set of elements and a law of combination, such that the result obtained by combining two elements of the group will itself be an element of the group. The precise definition contains a little more but the secret lies here.

Whatever the field of science may be and whatever the laws of change, one has but to discover a rule of operation on the abstract symbols in order that the details of the science may be matched in a one to one manner with the symbols.

Logically it should be possible to develop a mathematical system which has no possible use. It has appeared at times that such was the case but far more often the pure mathematics of one decade has become the handmaiden of science in the next.