

Integer-valued Equivalent Resistances

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Introductory physics course usually covers simple circuit theory and hence the student faces problems of finding the equivalent resistance of two resistors placed in parallel. The relationship is:

$$1/R_{eq} = 1/R_1 + 1/R_2 \quad \text{Eq.1}$$

The instructor's task is to find 'nice' values for R_1 , R_2 and R_{eq} such that the student doesn't need to waste time punching calculator keys. If the instructor restricts R_1 , R_2 and R_{eq} such that they are elements of the natural numbers, then the calculation is much simpler for the student. By restricting the resistor values in this way, we find that we have a diophantine equation. The problem of finding a general solution to the diophantine equation boils down to determining what conditions R_1 and R_2 must meet in order for R_{eq} to be a natural number. We have found a general solution to this problem.

Our solution can benefit the physics educator as well as the mathematician. For the introductory physics educator, our solution allows him to design assignment and test problems which illustrate the physics concepts with a minimum of calculation time. A quick search of introductory physics texts shows that the same 'nice' values are abused in example after example. Our solution allows an instructor to generate unfamiliar values which are still very 'nice.' To the math educator, our equation and solution offer a new problem in diophantine equations. In mathematics, classic diophantine problems stem from geometry. An example is the problem of finding integer valued solutions to the Pythagorean equation. Our equation offers the student a more tangible problem. Also, many of the key techniques in number theory are illustrated by our method of solution. With these benefits in mind, we turn to the solution.

We start by rewriting the equation form as such:

$$R_{eq} = R_1 R_2 / (R_1 + R_2) \quad \text{Eq.2}$$

Next, let D be the greatest common factor between R_1 and R_2 . We can write:

$$R_1 = DM \quad R_2 = DN \quad \text{Eq.3}$$

where M and N are the remaining factors of R_1 and R_2 respectively. We can see that M and N must be relatively prime, (greatest common factor equals 1) for otherwise, we could extract the common factor from M and N and make a further contribution to D . We substitute these expressions for R_1 and R_2 into Eq.2.

$$R_{eq} = DMN / (M + N) \quad \text{Eq.4}$$

For R_{eq} to be a natural number, then sum $(M + N)$ must evenly divide some combination of D , M , and N . Since we cannot extract a common factor from M and N , then

the sum $(M+N)$ cannot divide any combination which includes M or N . Therefore, the sum $(M+N)$ must divide D . So we can write:

$$D = k(M+N) \quad \text{Eq.5}$$

where k is some natural number. Now we substitute Eq.5 into Eq.3. After doing this we have:

$$\begin{aligned} R_1 &= kM(M+N) \\ R_2 &= kN(M+N) \\ R_{eq} &= kMN \end{aligned} \quad \text{Eq.6}$$

This is the general solution to our diophantine equation.

For any values of M and N such that M and N are relatively prime natural numbers, we can generate a solution to our equation. We now define a solution, where $k=1$, to be a primitive solution. Of course all positive integer multiples of a primitive solution are also solutions to the equation. We can show that each integer valued solution of Eq.1 can be generated by Eq.6 from exactly one combination of M and N .

PROOF:

Let M and N be relatively prime natural numbers. Likewise for M' and N' .

$$\begin{aligned} R_1 &= kM(M+N) & R_1' &= k'M'(M'+N') \\ R_2 &= kN(M+N) & R_2' &= k'N'(M'+N') \end{aligned} \quad \text{Eq.7}$$

If $R_1 = R_1'$ and $R_2 = R_2'$ then,

$$\begin{aligned} kM(M+N) &= k'M'(M'+N') \\ kN(M+N) &= k'N'(M'+N') \end{aligned} \quad \text{Eq.8}$$

Dividing equations, we get:

$$M/N = M'/N' \quad \text{Eq.9}$$

For this to be true we must have

$$M' = cM \text{ and } N' = cN \quad \text{Eq.10}$$

where c is a non-zero integer. Since M' and N' are relatively prime, then $c = 1$. Hence:

$$M = M' \text{ and } N = N' \quad \text{Eq.11}$$

which means, for a given R_1 and R_2 , only one combination of M and N will generate R_1 and R_2 .

From the form of our solution, M and N are indistinguishable and so we always choose $M < N$. M can equal N only if the case when $M = N = 1$. We extended our method of solution to an arbitrary number of parallel resistors.

An example of how to apply our method should make the solution clearer. Suppose the instructor wants to design a problem such that $R_{eq} = 30$ ohms. To start with, we find the prime factorization of 30.

$$30 = 2 \times 3 \times 5$$

Next, we start choosing values for k , M , and N from the prime factors. We don't neglect the possibility that k , M , or N could equal 1. Generally we start by choosing values for k first and then we sort through the remaining factors for values of M and N . Choices for M and N must meet the criterion that M and N be relatively prime and $M < N$, except for $M = N = 1$. We go through the choosing process systematically until we exhaust all combinations which meet the criteria. As an illustration of the method, the combinations for $R_{eq} = 30$ ohms are presented in Table 1. By following the procedure outlined above, the instructor can generate solutions for any value of R_{eq} .

TABLE 1. Combinations for $R_{eq} = 30$ Ohms

k	M	N	R_1	R_2
1	1	30	31	930
1	2	15	34	255
1	3	10	39	130
1	5	6	55	66
2	1	15	32	480
2	3	5	48	80
3	1	10	33	330
3	2	5	42	105
5	1	6	35	210
5	2	3	50	75
6	1	5	36	180
10	1	3	40	120
15	1	2	45	90
30	1	1	60	60

So far, our formulation has been restricted to positive integers, since ordinary resistances are never negative. However, with minor adjustments, we can extend our formulation to include negative integers, then we can look at another important topic covered in introductory physics courses, namely thin lenses. The thin lens equation is:

$$1/f = 1/i + 1/o \quad \text{Eq.12}$$

Here, the values for the image and object distances may also assume negative values by sign convention. It is easy to show that as soon as one chooses values for M and N when the image and object distances, the focal length, and the magnification have been determined. Namely:

$$\begin{aligned} o &= kM(M+N) \\ i &= kN(M+N) \\ f &= kMN \\ \text{Magnification} &= -M/N \end{aligned} \quad \text{Eq.13}$$

The fact that so many problem parameters are determined by choosing M and N points to the very nice feature of design simplicity offered by our method.

To conclude, our method succeeds in simplifying the task of designing introductory physics problems. Using the formula forms:

$$\begin{aligned} X &= kM(M+N) \\ Y &= kN(M+N) \\ Z &= kMN \end{aligned} \qquad \text{Eq.14}$$

where k , M , and N are non-zero integers and M and N are relatively prime, we can generate values of X , Y , and Z such that they will be non-zero integers. With this solution, the physics educator can design problems involving parallel resistors, series capacitors and thin lenses for introductory courses. To the math educator, our solution presents a fresh problem in diophantine equations which illustrates basic techniques of number theory analysis.